

# Normality of algebras over commutative rings, crossed pairs, and the Teichmüller class

Johannes Huebschmann

USTL, UFR de Mathématiques  
CNRS-UMR 8524  
Labex CEMPI (ANR-11-LABX-0007-01)  
59655 Villeneuve d'Ascq Cedex, France  
Johannes.Huebschmann@math.univ-lille1.fr

December 24, 2015

Dedicated to Ronnie Brown on the occasion of his 80th birthday.

## Abstract

Let  $S$  be a unitary commutative ring,  $Q$  a group that acts on  $S$  by ring automorphisms, and let  $R$  denote the subring of  $S$  that consists of the elements of  $S$  which are fixed under  $Q$ . A  $Q$ -normal  $S$ -algebra consists of a central  $S$ -algebra  $A$  and a homomorphism  $\sigma: Q \rightarrow \text{Out}(A)$  into the group  $\text{Out}(A)$  of outer automorphisms of  $A$  that lifts the action of  $Q$  on  $S$ . With respect to the abelian group  $U(S)$  of invertible elements of  $S$ , endowed with the  $Q$ -module structure coming from the  $Q$ -action on  $S$ , we associate to a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  a crossed 2-fold extension  $e_{(A, \sigma)}$  starting at  $U(S)$  and ending at  $Q$ , the *Teichmüller complex* of  $(A, \sigma)$  which, in turn, represents a class, the *Teichmüller class* of  $(A, \sigma)$ , in the third group cohomology group  $H^3(Q, U(S))$  of  $Q$  with coefficients in  $U(S)$ . Results are: (i) The Teichmüller class of a  $Q$ -normal algebra  $(A, \sigma)$  is zero if and only if the associated  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Aut}(M_{|Q|}(A))$  on the (suitably defined) matrix algebra  $M_{|Q|}(A)$  with entries from  $A$  is equivariant, i. e., arises from an action of  $Q$  by ring automorphisms on that matrix algebra. (ii) A central  $S$ -algebra  $A$  has a  $Q$ -normal structure with zero Teichmüller class if and only if  $A$  admits an embedding into an  $R$ -algebra  $C$  in such a way that certain compatibility conditions are satisfied which we refer to by the terminology *(strong) Deuring embedding*. Given such an embedding of a central  $S$ -algebra  $A$  into an  $R$ -algebra  $C$ , the compatibility conditions imply, in particular, that the embedding of  $A$  into  $C$  induces a  $Q$ -normal structure on  $A$ ; in the special case where  $A$  is an Azumaya  $S$ -algebra and  $S|R$  a Galois extension of commutative rings, the algebra  $C$  may be taken to be an Azumaya  $R$ -algebra. (iii) Suitable equivalence classes of  $Q$ -normal Azumaya  $S$ -algebras constitute a group  $\text{XB}(S, Q)$ , the *crossed Brauer group* of  $S$  relative to the  $Q$ -action, the assignment to a  $Q$ -normal Azumaya  $S$ -algebra of its Teichmüller class induces a homomorphism  $t: \text{XB}(S, Q) \rightarrow H^3(Q, U(S))$ , referred to as the *Teichmüller map*, and when  $Q$  is a finite group, the kernel of the Teichmüller map consists precisely of the classes of equivariant  $Q$ -normal  $S$ -algebras. (iv) To any member of a suitably defined abelian group  $k\mathcal{R}ep(Q, \mathcal{B}_{S, Q})$  of classes of representations of  $Q$  in the  $Q$ -graded Brauer category  $\mathcal{B}_{S, Q}$  of  $S$  with respect to the given action of  $Q$  on  $S$ , we associate a  $Q$ -normal algebra, hence a Teichmüller class, and thence the Teichmüller map is defined on  $k\mathcal{R}ep(Q, \mathcal{B}_{S, Q})$ . An obvious homomorphism from  $\text{XB}(S, Q)$  to  $k\mathcal{R}ep(Q, \mathcal{B}_{S, Q})$

is injective, and the Teichmüller map on  $\mathrm{XB}(S, Q)$  coincides with the composite of that injection with the Teichmüller map on  $k\mathcal{R}\mathrm{ep}(Q, \mathcal{B}_{S, Q})$ . The homomorphism from  $\mathrm{XB}(S, Q)$  to  $k\mathcal{R}\mathrm{ep}(Q, \mathcal{B}_{S, Q})$  is an isomorphism if the subgroup  $\kappa_Q(Q)$  of  $\mathrm{Aut}(S)$  is a finite group. (v) Given a suitably defined  $Q$ -normal Galois extension  $T|S$  of commutative rings, with structure extension  $N \twoheadrightarrow G \twoheadrightarrow Q$  of the group  $Q$  by the Galois group  $N = \mathrm{Aut}(T|S)$  of  $T|S$  and  $G$ -action on  $T$  by ring automorphisms, the abelian group  $\mathrm{Xpext}(G, N; \mathrm{U}(T))$  of congruence classes of crossed pairs relative to the group extension and the abelian group  $\mathrm{U}(T)$  of invertible elements of the ring  $T$ , endowed with the  $G$ -module structure coming from the  $G$ -action on  $T$ , is defined; now the assignment to a crossed pair  $(e, \psi)$  with respect to the data of the associated suitably defined crossed pair algebra  $(A_e, \sigma_\psi)$  yields a homomorphism from the abelian group  $\mathrm{Xpext}(G, N; \mathrm{U}(T))$  of congruence classes of crossed pairs relative to the data to the kernel  $\mathrm{XB}(T|S; G, Q)$  of the obvious homomorphism from  $\mathrm{XB}(S, Q)$  to  $\mathrm{XB}(T, G)$ . (vi) Under the circumstances of (v), a class  $k \in \mathrm{H}^3(Q, \mathrm{U}(S))$  is the Teichmüller class of some crossed pair algebra  $(A_e, \sigma_\psi)$  with respect to  $T|S$  if and only if, under inflation  $\mathrm{H}^3(Q, \mathrm{U}(S)) \rightarrow \mathrm{H}^3(G, \mathrm{U}(T))$ , the class  $k$  goes to zero. (vii) When the group  $Q$  is finite, the equivariant Picard group of  $S$  relative to  $Q$ , the  $Q$ -invariant subgroup  $\mathrm{Pic}(S)^Q$  of the ordinary Picard group  $\mathrm{Pic}(S)$  of  $S$ , the equivariant Brauer group of  $S$  relative to  $Q$ , the crossed Brauer group  $\mathrm{XB}(S, Q)$ , and the first, second, and third cohomology groups of  $Q$  with coefficients in  $\mathrm{U}(S)$  fit into a seven term exact sequence involving the Teichmüller map from  $\mathrm{XB}(S, Q)$  to  $\mathrm{H}^3(Q, \mathrm{U}(S))$ . (viii) Given a  $Q$ -normal Galois extension  $T|S$  of rings, with associated group extension  $\mathrm{Aut}(T|S) \twoheadrightarrow G \twoheadrightarrow Q$  and  $G$ -action on  $T$  by ring automorphisms, the various Picard groups, the obvious naive relative versions of the equivariant Brauer group and of the crossed Brauer group, the first, second, and third group cohomology groups of  $Q$  with coefficients in  $\mathrm{U}(S)$ , and the group cohomology group  $\mathrm{H}^3(G, \mathrm{U}(T))$  fit into an eight term exact sequence involving the Teichmüller map and the inflation map from  $\mathrm{H}^3(Q, \mathrm{U}(S))$  to  $\mathrm{H}^3(G, \mathrm{U}(T))$ . (ix) A more sophisticated variant of the relative theory yields similar seven and eight term exact sequences and behaves better with regard to comparison with group cohomology than does the naive relative theory.

2010 Mathematics Subject Classification: 12G05 13B05 16H05 16K50 16S35 20J06

Keywords and Phrases: Teichmüller cocycle, crossed module, crossed pair, normal algebra, crossed product, Deuring embedding problem, group cohomology, Galois theory of commutative rings, Azumaya algebra, Brauer group, Galois cohomology, non-commutative Galois theory, non-abelian cohomology

## Contents

|          |  |           |
|----------|--|-----------|
| <b>1</b> | <b>Preliminaries</b>   | <b>9</b>  |
| 1.1      | Basics . . . . .   | 9         |
| 1.2      | Galois extensions of commutative rings . . . . .   | 10        |
| 1.3      | Crossed modules . . . . .  | 10        |
| <b>2</b> | <b>Stably graded symmetric monoidal categories</b>   | <b>11</b> |
| 2.1      | Symmetric monoidal categories . . . . .  | 12        |
| 2.2      | The Brauer category associated to a commutative ring . . . . .   | 12        |
| 2.3      | Stably graded categories . . . . .   | 13        |
| 2.4      | Stably graded symmetric monoidal categories . . . . .  | 14        |
| 2.5      | The stably $Q$ -graded Brauer category associated to a commutative ring endowed with a $Q$ -action . . . . . | 16        |

|           |   |           |
|-----------|---|-----------|
| 2.6       | Picard categories . . . . .   | 17        |
| 2.7       | Change of actions . . . . .   | 18        |
| <b>3</b>  | <b>Normal algebras and their Teichmüller complexes</b>                          | <b>19</b> |
| 3.1       | Normal algebras . . . . .   | 19        |
| 3.2       | Discussion of normality and generalized normality . . . . .                     | 20        |
| 3.3       | Equivariant algebras and scalar extension . . . . .                             | 22        |
| 3.4       | The Teichmüller class of a $Q$ -normal algebra . . . . .                        | 22        |
| 3.5       | Opposite algebras . . . . .   | 23        |
| 3.6       | Matrix algebras . . . . .   | 23        |
| 3.7       | Tensor products . . . . .   | 24        |
| 3.8       | Behaviour under change of actions . . . . .                                     | 24        |
| 3.9       | Embedding algebras into algebras with smaller center . . . . .                  | 25        |
| 3.10      | The normal algebra associated to a generalized normal Azumaya algebra . . . . . | 27        |
| 3.11      | The Teichmüller class of a generalized $Q$ -normal Azumaya algebra . . . . .    | 34        |
| <b>4</b>  | <b>Crossed products with normal algebras</b>                                    | <b>34</b> |
| 4.1       | First crossed product algebra construction . . . . .                            | 35        |
| 4.2       | Second crossed product algebra construction . . . . .                           | 35        |
| 4.3       | Equivalence of the two crossed product algebra constructions . . . . .          | 35        |
| 4.4       | Properties of the crossed product algebra . . . . .                             | 36        |
| <b>5</b>  | <b>Normal algebras with zero Teichmüller class</b>                              | <b>39</b> |
| <b>6</b>  | <b>Normal ring extensions</b>   | <b>42</b> |
| <b>7</b>  | <b>Crossed pair algebras</b>  | <b>43</b> |
| 7.1       | Crossed pairs . . . . .   | 43        |
| 7.2       | Crossed pairs and normal algebras . . . . .                                     | 46        |
| <b>8</b>  | <b>Normal Deuring embedding and Galois descent for Teichmüller classes</b>      | <b>48</b> |
| 8.1       | The definitions . . . . .   | 48        |
| 8.2       | Discussion of the notion of normal Deuring embedding . . . . .                  | 51        |
| 8.3       | Results related with the two notions of normal Deuring embedding . . . . .      | 55        |
| <b>9</b>  | <b>Induced normal and equivariant structures</b>                                | <b>64</b> |
| 9.1       | Induced $Q$ -normal structures . . . . .  | 65        |
| 9.2       | Proof of the second assertion of Theorem 3.24 . . . . .                         | 66        |
| 9.3       | Induced $Q$ -equivariant structures . . . . .                                   | 67        |
| 9.4       | Induced $Q$ -equivariant structures and crossed product algebras . . . . .      | 68        |
| <b>10</b> | <b>Crossed Brauer group, generalized crossed Brauer group, and Picard group</b> | <b>69</b> |
| 10.1      | Crossed Brauer group . . . . .  | 69        |
| 10.2      | Crossed Brauer group and Picard group . . . . .                                 | 70        |
| 10.3      | The generalized crossed Brauer group . . . . .                                  | 73        |
| 10.4      | Behaviour under $Q$ -normal Galois extensions . . . . .                         | 73        |

|  |           |
|--|-----------|
| <b>11 The equivariant Brauer group</b>   | <b>75</b> |
| 11.1 The construction . . . . .  | 75        |
| 11.2 Some properties of the equivariant Brauer group . . . . .                                   | 75        |
| 11.3 Group extensions and equivariant Azumaya algebras . . . . .                                 | 77        |
| <b>12 The seven and eight term exact sequences</b>   | <b>77</b> |
| <b>13 Relationship with the eight term exact sequence in the cohomology of a group extension</b> | <b>81</b> |
| 13.1 Relationship between the two long exact sequences . . . . .                                 | 81        |
| 13.2 An application . . . . .  | 82        |
| 13.3 A variant of the relative theory . . . . .  | 83        |
| 13.3.1 The standard approach . . . . .   | 83        |
| 13.3.2 The Morita equivalence approach . . . . .   | 87        |
| <b>14 Examples</b>   | <b>90</b> |
| <b>15 Appendix</b>   | <b>90</b> |
| 15.1 Examples of symmetric monoidal categories . . . . .   | 90        |
| 15.2 Examples of stably $Q$ -graded symmetric monoidal categories . . . . .                      | 91        |
| 15.3 The standard constructions revisited . . . . .  | 92        |
| <b>References</b>  | <b>94</b> |

## Introduction

Let  $S$  be a unitary commutative ring,  $Q$  a group, and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of  $Q$  on  $S$  by ring automorphisms; we do not assume the group  $Q$  to be finite nor  $\kappa_Q$  to be injective. We refer to an algebra  $A$  having  $S$  as its center as a *central  $S$ -algebra*. Given a central  $S$ -algebra  $A$ , we denote by  $\text{Aut}(A)$  the group of ring automorphisms of  $A$  and by  $\text{Out}(A)$  that of ring automorphisms modulo inner ones. We *define* a  $Q$ -normal  $S$ -algebra to be a pair  $(A, \sigma)$ , where  $A$  is a central  $S$ -algebra and  $\sigma: Q \rightarrow \text{Out}(A)$  a homomorphism that lifts the action  $\kappa_Q$  of  $Q$  on  $S$  in the sense that the composite of  $\sigma$  with the obvious map  $\text{Out}(A) \rightarrow \text{Aut}(S)$  coincides with  $\kappa_Q$ . The special case where  $S$  is a field and  $Q$  a finite group of automorphisms of  $S$  is classical; in that case, a finite dimensional central simple  $S$ -algebra  $A$  having the property that each member of  $Q$  can be extended to an automorphism of  $A$  was termed  *$Q$ -normal* by Eilenberg and Mac Lane [EM48]. In view of the Skolem-Noether theorem, see, e. g., Proposition 3.1 below for a generalization, the restriction map  $\text{Out}(A) \rightarrow \text{Aut}(S)$  is then injective, and our definition is then therefore equivalent to that of Eilenberg and Mac Lane [EM48]. The classical case was studied already by Teichmüller [Tei40]. Teichmüller associated to a  $Q$ -normal central simple algebra over a field  $S$  a 3-cocycle of  $Q$  with values in the multiplicative group  $U(S)$  of non-zero elements of the field  $S$ , endowed with the  $Q$ -module structure coming from the  $Q$ -action on  $S$ ; this 3-cocycle was then termed the *Teichmüller-cocycle* by Eilenberg and Mac Lane [EM48], see also [Mac48b], [Mac48a]. For the case of a general commutative unitary ring  $S$  and a general action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$  by ring automorphisms, we shall extend the classical results of Teichmüller, Eilenberg-Mac Lane, and others.

Relative to the abelian group  $U(S)$  of invertible elements of the ring  $S$ , endowed with the  $Q$ -module structure coming from the  $Q$ -action on  $S$ , to any  $Q$ -normal  $S$ -algebra  $(A, \sigma)$ , we shall associate a crossed 2-fold extension  $e_{(A, \sigma)}$  starting at  $U(S)$  and ending at  $Q$ ; see Subsection 3.4 below for details. We then refer to this crossed 2-fold extension as the *Teichmüller complex* of  $(A, \sigma)$ . In view of the interpretation of the third group cohomology group  $H^3(Q, U(S))$  of  $Q$  with coefficients in  $U(S)$  in terms of crossed 2-fold extensions given in [Hue80], see also [Mac79],  $e_{(A, \sigma)}$  represents a class  $[e_{(A, \sigma)}] \in H^3(Q, U(S))$ , and we shall refer to this class as the *Teichmüller class* of  $(A, \sigma)$ . In Subsection 3.4 we shall show that, in the cocycle description, the Teichmüller class is the one represented by a Teichmüller cocycle.

Exploiting the description of the Teichmüller class in terms of the Teichmüller complex, we shall show how the classical results related with the Teichmüller cocycle for finite dimensional normal central simple algebras extend to general  $Q$ -normal algebras as defined above. We do not a priori assume  $A$  to be, e.g., an Azumaya algebra. We shall make such an assumption only when truly necessary (in Subsections 3.2, 3.10, 3.11, and from Section 10 on) but to begin with such an assumption would hide the formal nature of the arguments. We will now explain the results of the paper.

(1) Let  $(A, \sigma)$  be a  $Q$ -normal central  $S$ -algebra. Theorem 5.1 below says that  $(A, \sigma)$  *has zero Teichmüller class if and only if the  $Q$ -normal structure  $\sigma|_Q: Q \rightarrow \text{Out}(M_{|Q|}(A))$  induced from  $\sigma$  on the matrix algebra  $M_{|Q|}(A)$  over  $A$  is equivariant*, i.e., if and only if the  $Q$ -normal structure  $\sigma|_Q$  arises from an ordinary action of  $Q$  on  $M_{|Q|}(A)$  by ring automorphisms. In particular, if  $S|R$  is a Galois extension of commutative rings over  $R = S^Q$  with group  $Q$  then an equivariant structure on  $M_{|Q|}(A)$  comes from extension of scalars, by Galois descent, and so  $(A, \sigma)$  then *has zero Teichmüller class if and only if  $(M_{|Q|}(A), \sigma|_Q)$  arises by extension of scalars*.

(2) In order to explain our second result suppose for simplicity that  $S|R$  is a Galois extension of commutative rings over  $R = S^Q$  with Galois group  $Q$ . Corollary 5.6 below says that then *a central  $S$ -algebra  $A$  has a  $Q$ -normal structure  $\sigma$  with zero Teichmüller class if and only if  $A$  admits an embedding into a central  $R$ -algebra  $C$  so that (i) the centralizer of  $S$  in  $C$  is just  $A$ , and (ii) each automorphism  $\kappa_Q(q)$  of  $S$ , as  $q$  ranges over  $Q$ , extends to an inner automorphism  $\alpha$  of  $C$  which (in view of (i)) maps  $A$  to itself in such a way that the class of  $\alpha|_A$  in  $\text{Out}(A)$  coincides with  $\sigma(q)$ ; moreover, if  $A$  is an Azumaya  $S$ -algebra then  $C$  may be taken to be an Azumaya  $R$ -algebra*.

In the classical situation, such an embedding problem was raised by Deuring [Deu36], and Teichmüller apparently invented his cocycle in order to settle Deuring's embedding problem. Actually, in Theorem 5.4 below, we shall give a somewhat more general result than that just spelled out.

(3) Suitable equivalence classes of  $Q$ -normal Azumaya  $S$ -algebras constitute an abelian group, the *crossed Brauer group*, which we shall denote by  $\text{XB}(S, Q)$ , and the Teichmüller class depends only on the class in the crossed Brauer group, see Section 10 below. Moreover, the crossed Brauer group is a functor on a suitable category, to be explained in Subsection 2.7 below, the map  $t: \text{XB}(S, Q) \rightarrow H^3(Q, U(S))$  given by the assignment to a  $Q$ -normal Azumaya  $S$ -algebra of its Teichmüller complex is a natural homomorphism which we refer to henceforth as the *Teichmüller map*, and in case that  $Q$  is a finite group, the kernel of the Teichmüller map consists precisely of the classes of equivariant  $Q$ -normal  $S$ -algebras, in view of (1).

The results spelled out as (1), (2) and (3) extend the corresponding results of Teichmüller [Tei40]. For in the classical situation where  $Q$  is a finite group of automorphisms of a field  $S$ ,

the crossed Brauer group  $\text{XB}(S, Q)$  comes down to the subgroup  $\text{B}(S)^Q$  of the Brauer group  $\text{B}(S)$  of  $S$  that consists of the classes fixed under  $Q$ , and these are precisely the  $Q$ -normal classes.

(4) An alternate approach in terms of the “group-like stably  $Q$ -graded symmetric monoidal category”  $\mathcal{B}_{S,Q}$  associated to the symmetric monoidal *Brauer* category  $\mathcal{B}_S$  of the ground ring  $S$  and the  $Q$ -action on  $S$  [FW71b], [FW74], [FW00], see Section 2 for details, involves the abelian group  $k\mathcal{R}\text{ep}(Q, \mathcal{B}_{S,Q})$  of classes of representations of  $Q$  in  $\mathcal{B}_{S,Q}$  (see Subsection 2.5 for the notation), and we refer to the abelian group  $k\mathcal{R}\text{ep}(Q, \mathcal{B}_{S,Q})$  as the *generalized crossed Brauer group* and to its members as *generalized  $Q$ -normal Azumaya algebras*, for the following reason: In Subsection 3.10 below we shall associate, to any generalized  $Q$ -normal Azumaya algebra, an ordinary  $Q$ -normal algebra whose underlying algebra is an Azumaya algebra if and only if the subgroup  $\kappa_Q(Q)$  of  $\text{Aut}(S)$  is a finite group; see Theorem 3.19 for details. In Subsection 3.11 we then define the Teichmüller class of the given generalized  $Q$ -normal Azumaya algebra to be the Teichmüller class of the associated ordinary  $Q$ -normal algebra. Here our approach in terms of general algebras rather than just Azumaya algebras pays off since the  $Q$ -normal algebra associated to a generalized  $Q$ -normal Azumaya algebra need not be an Azumaya algebra. The map  $t: k\mathcal{R}\text{ep}(Q, \mathcal{B}_{S,Q}) \rightarrow \text{H}^3(Q, \text{U}(S))$  given by the assignment to a generalized  $Q$ -normal Azumaya  $S$ -algebra of its Teichmüller class is a natural homomorphism which, combined with the obvious homomorphism from the crossed Brauer group  $\text{XB}(S, Q)$  to  $k\mathcal{R}\text{ep}(Q, \mathcal{B}_{S,Q})$ , necessarily injective, cf. Theorem 10.10, coincides with the Teichmüller cocycle map from  $\text{XB}(S, Q)$  to  $\text{H}^3(Q, \text{U}(S))$  defined previously. If, furthermore, the subgroup  $\kappa_Q(Q)$  of  $\text{Aut}(S)$  is a finite group, the obvious homomorphism from the crossed Brauer group  $\text{XB}(S, Q)$  to  $k\mathcal{R}\text{ep}(Q, \mathcal{B}_{S,Q})$  is an isomorphism of abelian groups. See Subsection 10.3 for details. A byproduct of our reasoning is the observation that, with the notation  $\text{B}(S, Q)$  for the subgroup of the Brauer group  $\text{B}(S)$  whose members are represented by Azumaya  $S$ -algebras  $A$  having the property that each automorphism of the kind  $\kappa_Q(x)$  of  $S$  as  $x$  ranges over  $Q$  extends to an automorphism of  $A$ , the canonical homomorphism from  $\text{B}(S, Q)$  to  $\text{B}(S)^Q = \text{H}^0(Q, \text{B}(S))$  is an isomorphism when the subgroup  $\kappa_Q(Q)$  of  $\text{Aut}(S)$  is a finite group; see Corollary 3.20 for details.

(5) In Section 6, we shall introduce the concept of a  $Q$ -normal Galois extension of commutative rings; associated to such a  $Q$ -normal Galois extension  $T|S$  of commutative rings is a *structure extension*  $e_{(T|S)}: N \twoheadrightarrow G \twoheadrightarrow Q$  of  $Q$  by the Galois group  $N = \text{Aut}(T|S)$  of  $T|S$  and an action  $G \rightarrow \text{Aut}(T)$  of  $G$  on  $T$  by ring automorphisms. In Section 7, we shall associate to a *crossed pair*  $(e, \psi)$  with respect to  $e_{(T|S)}$  and  $\text{U}(T)$ , endowed with the  $G$ -module structure coming from the  $G$ -action on  $T$ , see [Hue81b] or Section 7 below for details on the crossed pair concept, a  $Q$ -normal crossed product algebra  $(A_e, \sigma_\psi)$  which we refer to as a *crossed pair algebra*. The crossed pair algebra  $(A_e, \sigma_\psi)$  represents a member of the kernel  $\text{XB}(T|S; G, Q)$  of the obvious homomorphism from  $\text{XB}(S, Q)$  to  $\text{XB}(T, G)$ ; this homomorphism exists and is unique, in view of the functoriality of the crossed Brauer group. Actually, the assignment to  $(e, \psi)$  of  $(A_e, \sigma_\psi)$  yields a natural homomorphism

$$\text{Xpext}(G, N; \text{U}(T)) \longrightarrow \text{XB}(T|S; G, Q) \quad (0.1)$$

of abelian groups from the corresponding abelian group  $\text{Xpext}(G, N; \text{U}(T))$  of congruence classes of crossed pairs introduced in [Hue81b] to the subgroup  $\text{XB}(T|S; G, Q)$  of the crossed Brauer group. Concerning the crossed pair concept, suffice it to mention at this stage that the notion of crossed pair generalizes that of crossed module.

(6) Let  $T|S$  be a  $Q$ -normal Galois extension of commutative rings, with structure extension  $e_{(T|S)}: N \twoheadrightarrow G \twoheadrightarrow Q$  of  $Q$  by  $N = \text{Aut}(T|S)$  and associated  $G$ -action on  $T$ . Theorem 7.5 below says that *a class  $k \in H^3(Q, U(S))$  is the Teichmüller class of some crossed pair algebra  $(A_e, \sigma_\psi)$  with respect to the data if and only if  $k$  is split in  $T|S$*  in the sense that, under inflation  $H^3(Q, U(S)) \rightarrow H^3(G, U(T))$ , the class  $k$  goes to zero. This yields a partial answer to the question as to which classes in  $H^3(Q, U(S))$  are Teichmüller classes.

Now, in the classical situation,  $T|S$  is a Galois field extension with Galois group  $Q$ , and we denote by  $B(T|S)$  the subgroup of the Brauer group  $B(S)$  of  $S$  that consists of the classes split by  $T$ ; the group  $\text{Xpext}(G, N; U(T))$  of crossed pair extensions now comes down to the subgroup  $H^2(N, U(T))^Q$  of  $H^2(N, U(T))$  that consists of the classes fixed by  $Q$ , the crossed Brauer group  $\text{XB}(T|S; G, Q)$  then boils down to the subgroup  $B(T|S)^Q$  of  $B(T|S)$  that consists of the classes fixed by  $Q$ —this is related with Speiser’s principal genus theorem (which is often referred to as Hilbert’s Satz 90)—, and the homomorphism (0.1) amounts to the classical isomorphism

$$H^2(N, U(T))^Q \longrightarrow B(T|S)^Q. \quad (0.2)$$

Hence (5) generalizes the corresponding result of Eilenberg and Mac Lane [EM48], and so does (6): For Eilenberg-Mac Lane’s characterization of the Teichmüller classes given in [EM48, Theorem 7.1, Theorem 10.1] is actually a crossed product theorem which covers all the Teichmüller classes since classically each  $Q$ -normal algebra class contains a  $Q$ -normal crossed product algebra, but such a result does not hold in the general situation considered here.

(7) With respect to the data, we shall denote the equivariant Brauer group by  $\text{EB}(S, Q)$ , see Section 11 below for details. For the particular case where the group  $Q$  is finite, the exact sequence (12.1) below involving the Teichmüller cocycle map  $t$  yields an extension of the kind

$$\dots \rightarrow (\text{Pic}(S))^Q \rightarrow H^2(Q, U(S)) \rightarrow \text{EB}(S, Q) \rightarrow \text{XB}(S, Q) \xrightarrow{t} H^3(Q, U(S))$$

of the corresponding classical low degree four term exact sequence by three more terms. If, furthermore,  $S|R$  is a Galois extension of commutative rings over  $R = S^Q$  with Galois group  $Q$ , by Galois descent, the abelian group  $\text{EB}(S, Q)$  is isomorphic to the ordinary Brauer group  $B(R)$  of  $R$ .

(8) Given a  $Q$ -normal Galois extension  $T|S$  of commutative rings with associated structure extension  $e_{(T|S)}: \text{Aut}(T|S) \twoheadrightarrow G \twoheadrightarrow Q$  and  $G$ -action on  $T$ , let  $\text{EB}(T|S; G, Q)$  denote the kernel of the induced homomorphism from  $\text{EB}(S, Q)$  to  $\text{XB}(T, G)$ ; the exact sequence (12.6) below involving the Teichmüller map  $t$  now yields an extension of the kind

$$\dots \rightarrow H^2(Q, U(S)) \rightarrow \text{EB}(T|S; G, Q) \rightarrow \text{XB}(T|S; G, Q) \xrightarrow{t} H^3(Q, U(S)) \xrightarrow{\text{inf}} H^3(G, U(T))$$

of the corresponding classical low degree four term exact sequence by four more terms. We will refer to the resulting theory as the *naive relative theory*. The exactness of that sequence at  $H^3(Q, U(S))$  follows from (4) above and from the naturality of the Teichmüller map. In Theorem 13.1 we shall compare that exact sequence with the eight term exact sequence in the cohomology of the group extension  $e_{(T|S)}$  with coefficients in  $U(T)$  constructed in [Hue81b].

(9) A more sophisticated variant of the relative theory involves certain abelian groups which we denote by  $B_{\text{fr}}(T|S)$ ,  $\text{EB}_{\text{fr}}(T|S; G, Q)$ ,  $\text{XB}_{\text{fr}}(T|S; G, Q)$ , and  $k\mathcal{R}ep(Q, \mathcal{B}_{T|S; G, Q})$ , see Subsection 13.3 below; these groups fits into seven and eight term exact sequences similar to those involving  $B(S)$ ,  $\text{EB}(S, Q)$ ,  $\text{XB}(S, Q)$ ,  $B(T|S)$ ,  $\text{EB}(T|S; G, Q)$ ,  $\text{XB}(T|S; G, Q)$ , and  $k\mathcal{R}ep(Q, \mathcal{B}_{S, Q})$ . This more sophisticated variant behaves better with regard to comparison of the theory with

group cohomology than does the naive relative theory; see Theorems 13.4–13.6 and Theorem 13.8 below.

In [Hue81a, Theorem 3 p. 303], we developed a conceptual description of the differential  $d_3: E_3^{0,2} \rightarrow E_3^{3,0}$  of the Lyndon-Hochschild-Serre spectral sequence associated to a group extension and a module over the extension group. In the case at hand, in view of [Hue81b, Subsection 1.4], the map  $\Delta$  lifts that differential to a map. Hence the exactness of (12.6) generalizes the result of Hochschild and Serre [HS53, p. 130] saying that, in the classical case, the Teichmüller map coincides with the transgression map in the low degree five term exact sequence in the cohomology of the corresponding group extension.

Generalizations of the Teichmüller cocycle map were also developed by Childs [Chi72], Fröhlich and Wall [FW00], Knus [Knu75], Pareigis [Par64], Ulbrich [Ul89], [Ul94], and Zelinski [Zel76]. Our approach is substantially different from that in each of those papers. Indeed, in those articles except [Knu75] and [FW00], given an action  $\kappa: Q \rightarrow \text{Aut}(S)$  of a group  $Q$  on the commutative ring  $S$ , an  $S$ -Azumaya algebra  $A$  is defined to be  $Q$ -normal if each automorphism  $\kappa(q)$  of  $S$ , as  $q$  ranges over  $Q$ , extends to an automorphism of  $A$ ; accordingly, the values of the Teichmüller cocycle map developed in those articles do not necessarily lie in the cohomology group  $H^3(Q, U(S))$ . When  $S$  is a field, in view of the Skolem-Noether theorem, that definition is equivalent to ours but over a general ring  $S$  this is not the case; needless, perhaps, to point out that, given the  $S$ -algebra  $A$ , with our definition involving a homomorphism  $\sigma: Q \rightarrow \text{Out}(A)$ , we indeed obtain a Teichmüller cocycle map with values in  $H^3(Q, U(S))$ . The (unpublished) manuscript [Knu75] offers a purely Amitsur complex approach. Over a general commutative ring  $S$ , (without a name,) the crossed Brauer group is introduced in [FW71b, p. 43], see also [FW00, Section 3], denoted there by  $QB(R, \Gamma)$  (beware: that notation  $Q$  has nothing to do with our notation  $Q$  for a group) where  $R$  corresponds to our notation  $S$  and  $\Gamma$  to our notation  $Q$ . While, in none of the papers by Fröhlich and Wall could I find a reference to Teichmüller nor to any of the follow-up papers thereof, in [FW00, Theorem 3.4(i)], I eventually discovered an explicit cocycle description of the Teichmüller cocycle map (without a name) from the crossed Brauer group  $XB(S, Q)$  to the corresponding third group cohomology group, as well as a cocycle description of the Teichmüller cocycle map (still without a name) from  $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$  to the corresponding third group cohomology group. Also the injective homomorphism from  $XB(S, Q)$  to  $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$  is given in [FW00], as well as a proof of the fact that, for finite  $Q$ , this homomorphism is an isomorphism. Indeed, our construction of the  $Q$ -normal algebra associated to a generalized  $Q$ -normal Azumaya algebra given in Subsection 3.10 below involves a variant of a construction that is used in [FW00] to establish the surjectivity of the homomorphism from  $XB(S, Q)$  to  $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$  in the special case where the group  $Q$  is finite. However, to my knowledge, in the literature, the Teichmüller cocycle and its offspring was never related with the crossed pair concept nor with an equivalent notion. It is, perhaps, also worthwhile noting that none of the ten articles that, according to the MR citations, quote [FW00], contains a reference to Teichmüller nor to any of the follow-up papers. Likewise, among the articles that, according to the MR citations, quote Teichmüller's original paper [Tei40], only [CGG00] and [CGO01] contain a reference to the paper [FW74] by Fröhlich and Wall, and there is no other reference to the papers by Fröhlich and Wall related with the subject, in particular, no reference at all to [FW00] where a version of the Teichmüller cocycle appears. The crossed pair concept is closely related with that of a pseudo-module [Tay53]; that paper arose out of a thesis supervised by J.H.C. Whitehead, pseudo-modules being a generalization of crossed modules. It seems that, thereafter, pseudo-modules were largely forgotten. In Remark 7.3 below we



shall make the relationship between crossed pairs and pseudo-modules explicit. We also note that, in class field theory, over a field, the Teichmüller cocycle yields a certain obstruction (passage from local to global) [Nak53] but is rarely spelled out explicitly; an explicit hint may be found, e. g., in [Tat67, Section 11 p. 199].

In this paper, we shall work over commutative rings (rather than fields on one hand or schemes on the other), and our methods are conceptual, do not involve chain complexes, and avoid cocycle calculations. It might be worthwhile to extend our methods to ringed spaces, cf. [Aus66].

I am indebted to the referee for a number of valuable comments. It is a pleasure to dedicate this paper to Ronnie Brown. In my thesis, written with the help and encouragement of B. Eckmann, I had developed an interpretation of the group cohomology groups in terms of crossed  $n$ -fold extensions, cf. [Hue80]. At the time (in 1977), S. Mac Lane had suggested I should get in contact with R. Brown which I did, and Ronnie got excited seeing that interpretation of group cohomology. It is fair to say, if I hadn't met Ronnie at the time I might not have become a research mathematician. Needless to point out, Ronnie has a long record of research papers dealing with crossed modules, crossed  $n$ -fold extensions and variants thereof, as well as numerous articles on applications of these notions in topology and on foundational issues related with these notions. The recent monograph [BHS11] reflects this activity and contains a host of references. In the present paper we show how crossed modules and crossed pairs, a somewhat more general notion, occur elsewhere in mathematical nature, in a way that is, perhaps, a bit surprising at first glance.

## 1 Preliminaries

### 1.1 Basics

The notation employed in the introduction remains fixed throughout the paper. For the reader's convenience we recall some basic definitions and facts used in the paper.

Below we will often write the identity morphism on an object as 1. Given a morphism  $\alpha$  on an object, we will occasionally denote the restriction to a subobject by " $\alpha|$ ". Given an action  $\kappa: G \rightarrow \text{Aut}_C(C)$  of a group  $G$  on an object  $C$  of a category  $C$ , we will write the action as

$$G \times C \longrightarrow C, (x, y) \longmapsto {}^x y, x \in G, y \in C.$$

The commutative unitary ring  $S$  remains fixed, unless the contrary is admitted explicitly, and the notation  $\otimes$  refers to the tensor product over  $S$ . By an  $S$ -algebra we mean an algebra  $A$  whose center contains  $S$  as a subring; thus given an  $S$ -algebra  $A$ ,  $S$  acts faithfully on  $A$ . As in the introduction, a *central*  $S$ -algebra is an  $S$ -algebra such that  $S$  coincides with the center of  $A$ . Given an  $S$ -algebra  $A$  (not necessarily central), we denote by  $A^{\text{op}}$  the *opposite* algebra. Given the  $S$ -algebra  $A$ , consider the tensor product algebra  $A \otimes A^{\text{op}}$ ; the map

$$\eta: A \otimes A^{\text{op}} \rightarrow \text{Hom}_S(A, A), \eta(a \otimes b^{\text{op}})c = acb, a, b, c \in A, \quad (1.1)$$

turns  $A$  into an  $(A \otimes A^{\text{op}})$ -module. As in [AG60a], we refer to an  $S$ -algebra  $A$  as being *separable* when  $A$  is a projective as an  $(A \otimes A^{\text{op}})$ -module. A central separable  $S$ -algebra  $A$  is also referred to as an *Azumaya  $S$ -algebra*. This agrees with the definition in [Bou61, p. 180] since, by [AG60a, Theorem 2.1], a central  $S$ -algebra  $A$  is separable if and only if the map  $\eta$  is an isomorphism and if, as an  $S$ -module,  $A$  is finitely generated and projective.

Given an action of a group  $G$  on a (not necessarily commutative) ring  $\Lambda$ , written as  $(x, t) \mapsto {}^x t$ , for  $x \in G$  and  $t \in \Lambda$ , the *twisted group ring*  $\Lambda^t G$  has as its underlying  $\Lambda$ -module the free left  $\Lambda$ -module having as its basis the elements of  $G$ , with multiplication given by  $sxt y = s({}^x t)xy$ , where  $s, t \in \Lambda$ ,  $x, y \in G$ .

## 1.2 Galois extensions of commutative rings

As in the introduction, given a group  $Q$  and an action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$ , relative to the action, the twisted group ring  $S^t Q$  is defined. Using the notation  $R = S^Q$  for the subring of  $S$  that consists of all members of  $S$  left fixed by every element of  $Q$ , consider map

$$j: S^t Q \longrightarrow \text{End}_R(S)$$

given by  $j(sq)x = s({}^q x)$ ,  $s, x \in S$ ,  $q \in Q$ .

Let  $Q$  be a finite group of operators on the commutative ring  $S$  and write  $R = S^Q$ . Under these circumstances,  $S|R$  is a *Galois extension of commutative rings with Galois group*  $Q$  if any of the subsequent equivalent conditions (i)–(iii) holds:

(i) The ring  $S$  is a finitely generated projective  $R$ -module, and  $j$  is an isomorphism.

(ii) (*Galois descent*). Given a left  $S^t Q$ -module  $M$ , viewed as a left  $Q$ -module in the obvious way, the map

$$w: S \otimes_R M^Q \longrightarrow M, \quad w(s \otimes m) = sm, \quad s \in S, \quad m \in M^Q,$$

is an  $S$ -module isomorphism, indeed, an  $S^t Q$ -module isomorphism relative to the obvious  $S^t Q$ -module structure on  $S \otimes_R M^Q$ .

(iii) Given a member  $q$  of  $Q$  distinct from the neutral element and a maximal ideal  $\mathfrak{p}$  of  $S$ , there exists  $s = s(\mathfrak{p}, q)$  in  $S$  with  $s - {}^q s$  not in  $\mathfrak{p}$ .

The equivalence of (i)–(iii) is established in [CHR65, Theorem 1.3 p. 4]. In [AG60a, p. 396], (i) is taken as the definition of a Galois extension of commutative rings with Galois group  $Q$ .

**Example 1.1.** Given a Galois extension  $K|k$  of algebraic number fields with Galois group  $Q$ , the extension  $S|R$  of the associated rings of integers is a Galois extension of commutative rings with Galois group  $Q$  if and only if the extension is unramified [CHR65, Remark 1.5 (d) p. 7].

**Example 1.2.** Given a finite group  $Q$  of operators on a Hausdorff space  $Y$  with orbit space  $X$ , the ring extension  $C^0(Y)|C^0(X)$  of the associated rings of continuous functions is a Galois extension of commutative rings with Galois group  $Q$  if and only if the projection map  $Y \rightarrow X$  is an ordinary covering map [CHR65, Remark 1.5 (e) p. 7].

## 1.3 Crossed modules

A *crossed module*  $(C, \Gamma, \partial)$  [Whi49, p. 453] consists of groups  $C$  and  $\Gamma$ , an action of  $\Gamma$  on  $C$  (from the left), written as  $(\gamma, x) \mapsto {}^\gamma x$ ,  $\gamma \in \Gamma, x \in C$ , and a homomorphism  $\partial: C \rightarrow \Gamma$  of  $\Gamma$ -groups where  $\Gamma$  acts on itself by conjugation, subject to the axiom

$$bcb^{-1} = {}^{\partial b} c, \quad b, c \in C.$$

Given two crossed modules  $(C, \Gamma, \partial)$  and  $(C', \Gamma', \partial')$ , a *morphism*

$$(\varphi, \psi): (C, \Gamma, \partial) \longrightarrow (C', \Gamma', \partial')$$

of crossed modules is given by a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\partial} & \Gamma \\ \varphi \downarrow & & \downarrow \psi \\ C' & \xrightarrow{\partial'} & \Gamma' \end{array}$$

in the category of  $\Gamma$ -groups, the  $\Gamma$ -action on  $C'$  and  $\Gamma'$  being induced by the homomorphism  $\psi$ . A *crossed 2-fold extension* is an exact sequence of groups

$$e^2: 0 \longrightarrow M \longrightarrow C \xrightarrow{\partial} \Gamma \longrightarrow G \longrightarrow 1$$

involving a crossed module  $(C, \Gamma, \partial)$  [Hue80]; since  $M$  is then central in  $C$  it is an abelian group, and the  $\Gamma$ -action on  $C$  induces a  $G$ -module structure on  $M$ . Given two crossed 2-fold extensions

$$e^2: 0 \rightarrow M \rightarrow C \xrightarrow{\partial} \Gamma \rightarrow G \rightarrow 1, \quad \hat{e}^2: 0 \rightarrow \hat{M} \rightarrow \hat{C} \xrightarrow{\hat{\partial}} \hat{\Gamma} \rightarrow \hat{G} \rightarrow 1,$$

a *morphism*  $(\alpha, \varphi, \psi, \beta): e^2 \rightarrow \hat{e}^2$  of crossed 2-fold extensions is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & C & \xrightarrow{\partial} & \Gamma & \longrightarrow & G & \longrightarrow & 1 \\ & & \alpha \downarrow & & \varphi \downarrow & & \psi \downarrow & & \beta \downarrow & & \\ 0 & \longrightarrow & \hat{M} & \longrightarrow & \hat{C} & \xrightarrow{\hat{\partial}} & \hat{\Gamma} & \longrightarrow & \hat{G} & \longrightarrow & 1 \end{array}$$

in the category of groups such that  $(\varphi, \psi): (C, \Gamma, \partial) \rightarrow (\hat{C}, \hat{\Gamma}, \hat{\partial})$  is a morphism of crossed modules; a morphism of crossed 2-fold extensions of the kind  $(1, \varphi, \psi, 1): e^2 \rightarrow \hat{e}^2$  (so that, in particular,  $\hat{M} = M$  and  $\hat{G} = G$ ) is referred to as a *congruence*. The notion of congruence of group extensions with abelian kernel is classical, cf. [Mac67, IV.3 p. 109].

When the group  $G$  and the  $G$ -module  $M$  are fixed, under the equivalence relation generated by congruence, the classes of crossed 2-fold extensions starting at  $M$  and ending at  $G$  constitute an abelian group  $\text{Opext}^2(G, M)$ , and this group is naturally isomorphic to the ordinary group cohomology group  $H^3(G, M)$ ; this fact is a special case of the main result in [Hue80, § 7]. See also Mac Lane's Historical Note [Mac79].

## 2 Stably graded symmetric monoidal categories

At the referee's request, we will now recall some material from the theory of stably graded symmetric monoidal categories [FW71b], [FW74], [FW00] and we will, in particular, explain the significance of group-like stably graded symmetric monoidal categories for our purposes. This enables us to introduce notation needed thereafter in the paper.

## 2.1 Symmetric monoidal categories

A *symmetric monoidal category* [Mac71] consists of a category  $\mathcal{C}$ , a covariant functor  $\odot: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , an object  $E$  of  $\mathcal{C}$ , referred to as the *unit object*, and natural structure equivalences written as  $e: E \odot \cdot \rightarrow \cdot$ ,  $a: \cdot \odot (\cdot \odot \cdot) \rightarrow (\cdot \odot \cdot) \odot \cdot$ , and  $c_{A,B}: A \odot B \rightarrow B \odot A$ , where  $A$  and  $B$  range over objects of  $\mathcal{C}$ ; these pieces of structure are subject to certain nowadays standard axioms. The terminology in [FW71b], [FW74], [FW00] is ‘monoidal’ for the present ‘symmetric monoidal’. We now recall some more terminology from [op. cit.].

Consider a symmetric monoidal category  $\mathcal{C}$ . The notation  $k\mathcal{C}$  refers to the abelian monoid (with unit) of isomorphism classes of objects of  $\mathcal{C}$ , usually required to be a set, and  $U(\mathcal{C})$  denotes the group of automorphisms  $\text{Aut}_{\mathcal{C}}(E)$  of the unit object  $E$  in  $\mathcal{C}$  [FW74, §2 p. 235].

Given an object  $C$  of  $\mathcal{C}$ , for  $u \in U(\mathcal{C})$ , the composite

$$\vartheta_C(u): C \xrightarrow{e_C^{-1}} E \odot C \xrightarrow{u \odot 1_C} E \odot C \xrightarrow{e_C} C \quad (2.1)$$

defines an automorphism of  $C$  and, by [FW74, Theorem 2.2 p. 235],

$$\vartheta_C: U(\mathcal{C}) \longrightarrow \text{Aut}_{\mathcal{C}}(C) \quad (2.2)$$

is a homomorphism, an isomorphism if  $C$  is an invertible object of  $\mathcal{C}$  and, in general, the subgroup  $\text{im}(\vartheta_C) \subseteq \text{Aut}_{\mathcal{C}}(C)$  lies in the center of  $\text{Aut}_{\mathcal{C}}(C)$ . An object  $C$  of  $\mathcal{C}$  is *faithful* if  $\vartheta_C$  is injective [FW71b, §6 p. 47], [FW74, §4 p. 243].

A symmetric monoidal category  $\mathcal{C}$  is *group-like* if every object and every morphism in  $\mathcal{C}$  is invertible [FW74, §2 p. 237], [FW00, §1]. In the last reference, the definition of being group-like is applied only to “precise” symmetric monoidal categories but this is presumably not intended and, after all, in [FW74], the definition of being group-like is applied to general symmetric monoidal categories. The monoid  $k\mathcal{C}$  associated to a group-like symmetric monoidal category  $\mathcal{C}$  is an abelian group.

## 2.2 The Brauer category associated to a commutative ring

For later reference, we recall the *standard construction* of the *Brauer group*  $B(S)$  of the commutative ring  $S$  [AG60a, p. 381]. Two Azumaya  $S$ -algebras  $A_1$  and  $A_2$  are *Brauer equivalent* if there are faithful finitely generated projective  $S$ -modules  $M_1$  and  $M_2$  such that the Azumaya  $S$ -algebras  $A_1 \otimes \text{End}_S(M_1)$  and  $A_2 \otimes \text{End}_S(M_2)$  are isomorphic  $S$ -algebras. Given two faithful finitely generated projective  $S$ -modules  $M_1$  and  $M_2$ , the tensor product  $M_1 \otimes M_2$  is again a faithful finitely generated projective  $S$ -module, and the  $S$ -algebra  $\text{End}_S(M_1) \otimes \text{End}_S(M_2)$  is canonically isomorphic to the  $S$ -algebra  $\text{End}_S(M_1 \otimes M_2)$ . Hence that relation is indeed an equivalence relation, cf. [AG60a, p. 381], referred to as *Brauer equivalence*; under the operation of tensor product and under the assignment to the class of an Azumaya  $S$ -algebra  $A$  of the class of its opposite algebra  $A^{\text{op}}$ , the equivalence classes constitute an abelian group, the *Brauer group*  $B(S)$  of  $S$ , having the class of  $S$  as its unit element. The assignment to a commutative ring of its Brauer group is a functor from commutative ring to abelian groups. In particular, the  $Q$ -action on  $S$  induces a  $Q$ -action on the Brauer group  $B(S)$  of  $S$  that turns  $B(S)$  into a  $Q$ -module. Given a homomorphism  $f: S \rightarrow T$  of commutative rings, we denote by  $B(T|S)$  the kernel of the induced homomorphism  $B(S) \rightarrow B(T)$ . The abelian group  $B(T|S)$  really depends on the homomorphism  $f$  rather than just on  $S$  and  $T$  but, for intelligibility, we will stick to the familiar notation  $B(T|S)$  which is classical when  $S$  and  $T$  are fields (and  $f$  necessarily injective).

Recall that, given two Azumaya  $S$ -algebras  $A$  and  $B$ , a  $(B, A)$ -bimodule  $M$  is *invertible* if there exists an  $(A, B)$ -bimodule  $M'$  such that  $M \otimes_A M' \cong B$  as  $(B, B)$ -bimodules and  $M' \otimes_B M \cong A$  as  $(A, A)$ -bimodules, cf., e. g., [Bas68]. The *Brauer category*  $\mathcal{B}_S$  of the commutative ring  $S$  [FW71b, §3 p. 23], [FW74, p. 230], [FW00, §2], written in [FW71b, §3 p. 23] and [FW00, §2] as  $\mathcal{B}_R$  and in [FW74, p. 230] as  $\mathcal{B}r_R$ , has as *objects* the Azumaya  $S$ -algebras, a *morphism*  $[M]: A \rightarrow B$  in  $\mathcal{B}_S$  between two Azumaya algebras  $A$  and  $B$ , necessarily an isomorphism in  $\mathcal{B}_S$ , being an isomorphism class of an invertible  $(B, A)$ -bimodule  $M$ . Given three Azumaya  $S$ -algebras  $A, B, C$  and morphisms  $[{}_B M_A]: A \rightarrow B$  and  $[{}_A M_C]: C \rightarrow A$  in  $\mathcal{B}_S$ , the composition of  $[{}_B M_A]: A \rightarrow B$  with  $[{}_A M_C]: C \rightarrow A$  in  $\mathcal{B}_S$  is given by the morphism  $[{}_B M_A \otimes_A {}_A M_C]: C \rightarrow B$  in  $\mathcal{B}_S$ , where  $[{}_B M_A \otimes_A {}_A M_C]$  refers to the isomorphism class of the invertible  $(B, C)$ -bimodule  ${}_B M_A \otimes_A {}_A M_C$ . The operation of tensor product over the ground ring  $S$  and the assignment to an Azumaya  $S$ -algebra  $A$  of its opposite algebra  $A^{\text{op}}$  turn  $\mathcal{B}_S$  into a group-like symmetric monoidal category [FW74, §5 p. 247], [FW00, §2] having  $\text{U}(\mathcal{B}_S) = \text{Pic}(S)$  [FW00, §3]. The members of the abelian group  $k\mathcal{B}_S$  are Morita equivalence classes of Azumaya  $S$ -algebras. Since Morita equivalence is equivalent to Brauer equivalence, cf., e. g., [VZ78, p. 41], the canonical homomorphism from  $\text{B}(S)$  to the abelian group  $k\mathcal{B}_S$  is an isomorphism. By construction, then, given an Azumaya  $S$ -algebra  $A$ , its group  $\text{Aut}_{\mathcal{B}_S}(A)$  of automorphisms in  $\mathcal{B}_S$  is the group of faithful projective rank one  $(A, A)$ -bimodules, and the assignment to a projective rank one  $S$ -module  $J$  of the  $(A, A)$ -bimodule  $A \otimes J$  yields an isomorphism  $\text{Pic}(S) \rightarrow \text{Aut}_{\mathcal{B}_S}(A)$  of abelian groups.

## 2.3 Stably graded categories

As before,  $Q$  denotes a group. We view  $Q$  as a category with a single object. A  $Q$ -graded category is a pair  $(C_Q, g)$  that consists of a category  $C_Q$  and a functor  $g: C_Q \rightarrow Q$ , the *grading* [FW71b, §1 p. 2], [FW74, §3 p. 240], [FW00, §1]. The grading  $g$  of a  $Q$ -graded category  $(C_Q, g)$  is *stable* if, given an object  $C$  of  $C_Q$  and  $x \in Q$ , there is an equivalence  $f$  in  $C_Q$  with domain  $C$  and  $g(f) = x \in Q$  [FW71b, §1 p. 3], [FW74, §3 p. 240], [FW00, §1].

We will usually suppress the functor  $g$  from the notation unless it is convenient to spell it out for clarity. Given a  $Q$ -graded category  $C_Q$ , the notation  $\mathcal{K}er(C_Q)$  refers to the category having the same objects as  $C_Q$  but whose morphisms are only those of grade 1 [FW71b, §1 p. 2], [FW74, §3 p. 240], [FW00, §1]. Below we will always indicate the fact that a  $Q$ -graded category is under discussion by the subscript  $-_Q$ . Given a  $Q$ -graded category  $C_Q$ , we will then use the notation  $\mathcal{C}$  for  $\mathcal{K}er(C_Q)$ .

Given a stably  $Q$ -graded category  $C_Q$ , the group  $Q$  acts on  $k\mathcal{C} = k\mathcal{K}er(C_Q)$  as follows [FW71b, §1 p. 3], [FW74, Lemma 3.1 p. 240], [FW00, §1]: Given an object  $C$  of  $\mathcal{C} = \mathcal{K}er(C_Q)$  and  $x \in Q$ , keeping in mind that  $\mathcal{K}er(C_Q)$  and  $C_Q$  have the same objects, choose a morphism  $f: C \rightarrow D$  in  $C_Q$  of grade  $x$  and define the result of the action  ${}^x[C]$  of  $x$  on  $[C] \in k\mathcal{C}$  by  ${}^x[C] = [D] \in k\mathcal{C}$ . This action is well defined, cf. [FW74, §3 pp. 240/41]. Denote  $k\mathcal{C}$ , endowed with this  $Q$ -action, by  $k_Q\mathcal{C}_Q$ , cf. the notation  $k_\Gamma$  in [FW74, Lemma 3.1 p. 240]. The association  $C_Q \mapsto k_Q\mathcal{C}_Q$ , as  $C_Q$  ranges over stably  $Q$ -graded symmetric monoidal categories, defines a functor  $k_Q$  from the category of stably  $Q$ -graded symmetric monoidal categories to the category  $Q\text{Set}$  of  $Q$ -sets, cf. [FW74, Lemma 3.1 p. 240].

A  $Q$ -functor [FW74, §3 p. 241] is one over the identity map of  $Q$  or, equivalently, a functor preserving grades of morphisms. Below *natural transformation of  $Q$ -functors* will be required to be of grade 1. The *category  $\mathcal{R}ep(Q, C_Q)$  of representations of  $Q$  in a  $Q$ -graded category  $(C_Q, g)$*  is the category of  $Q$ -functors  $F: Q \rightarrow C_Q$ , that is, functors  $F$  from  $Q$  to  $C_Q$  such that

the composite  $g \circ F$  of  $F$  with the grade functor  $g$  is the identity functor on  $Q$ , and natural transformations of grade 1 [FW71b, §1 p. 2], [FW74, Introduction, §3 p. 241], [FW00, §1]. Thus an object of  $\mathcal{R}ep(Q, \mathcal{C}_Q)$  is a representation  $h: Q \rightarrow \text{Aut}_{\mathcal{C}_Q}(A)$ , in the category  $\mathcal{C}_Q$ , of  $Q$  on an object  $A$  of the category  $\mathcal{C} = \mathcal{K}er(\mathcal{C}_Q)$ .

Given two  $Q$ -graded categories  $(\mathcal{C}_Q, g)$  and  $(\mathcal{C}'_Q, g')$ , write  $\mathcal{C}_Q \times_Q \mathcal{C}'_Q$  for the *pull back category* of  $(g, g')$ . This pull back category acquires an obvious  $Q$ -grading, and this grading is stable if  $g$  and  $g'$  are. The pull back category yields the *product* in the category of  $Q$ -graded categories.

## 2.4 Stably graded symmetric monoidal categories

A *stably  $Q$ -graded symmetric monoidal category* [FW74, §3 p. 240], [FW00, §1] consists of a stably  $Q$ -graded category  $(\mathcal{C}_Q, g)$ , a covariant  $Q$ -functor  $\odot: \mathcal{C}_Q \times_Q \mathcal{C}_Q \rightarrow \mathcal{C}_Q$ , that is, the composite  $h_1 \odot h_2$  of two morphisms  $h_1$  and  $h_2$  is defined only when  $h_1$  and  $h_2$  have the same grade, and  $g(h_1 \odot h_2) = g(h_1) = g(h_2)$ , a covariant  $Q$ -functor  $E: Q \rightarrow \mathcal{C}_Q$ , referred to as the *unit object* of  $\mathcal{C}_Q$ , and grade 1 natural equivalences  $e: E \odot \cdot \rightarrow \cdot$ ,  $a: \cdot \odot (\cdot \odot \cdot) \rightarrow (\cdot \odot \cdot) \odot \cdot$ , and  $c_{A,B}: A \odot B \rightarrow B \odot A$ , where  $A$  and  $B$  range over objects of  $\mathcal{C}_Q$ ; these equivalences are subject to the standard axioms.

Let  $\mathcal{C}_Q$  be a stably  $Q$ -graded symmetric monoidal category. The category  $\mathcal{C} = \mathcal{K}er(\mathcal{C}_Q)$  acquires an obvious symmetric monoidal category structure having, in particular,  $E(e)$  as its unit object. (N.B.:  $E$  is a  $Q$ -functor and  $e \in Q$  refers to the neutral element of  $Q$ .) By definition, the *unit group*  $U(\mathcal{C}_Q)$  of  $\mathcal{C}_Q$  is the (abelian) unit group

$$U(\mathcal{C}) = U(\mathcal{K}er(\mathcal{C}_Q)) = \text{Aut}_{\mathcal{C}}(E(e))$$

of the category  $\mathcal{C} = \mathcal{K}er(\mathcal{C}_Q)$  [FW71b, (2.4) p. 8], [FW74, §3 p. 241], [FW00, §1].

The grading induces a surjective homomorphism  $\text{Aut}_{\mathcal{C}_Q}(E(e)) \rightarrow Q$  which fits into a group extension

$$e_{\mathcal{C}_Q}^{U(\mathcal{C})}: 1 \longrightarrow U(\mathcal{C}) \longrightarrow \text{Aut}_{\mathcal{C}_Q}(E(e)) \longrightarrow Q \longrightarrow 1, \quad (2.3)$$

and  $E: Q \rightarrow \text{Aut}_{\mathcal{C}_Q}(E(e))$  splits this group extension. Thus, via the  $Q$ -functor  $E$ , the group  $\text{Aut}_{\mathcal{C}_Q}(E(e))$  decomposes as the semi-direct product

$$\text{Aut}_{\mathcal{C}_Q}(E(e)) = U(\mathcal{C}) \rtimes Q, \quad (2.4)$$

and this decomposition induces a  $Q$ -module structure on  $U(\mathcal{C})$  [FW71b, §2 p. 8], [FW74, §3 p. 241], [FW00, §1]. We will use the notation  $U(\mathcal{C}_Q)$  for  $U(\mathcal{C})$ , endowed with the  $Q$ -module structure just explained.

Given an object  $C$  of  $\mathcal{C}_Q$ , the group  $\text{Aut}_{\mathcal{C}_Q}(C)$  of automorphisms of  $C$  in  $\mathcal{C}_Q$  has the subgroup of grade 1 automorphisms as a normal subgroup, this subgroup is canonically isomorphic to the group  $\text{Aut}_{\mathcal{C}}(C)$  of automorphisms of  $C$  in  $\mathcal{C} = \mathcal{K}er(\mathcal{C}_Q)$ , and the above homomorphism  $\vartheta_C: U(\mathcal{C}) \longrightarrow \text{Aut}_{\mathcal{C}}(C)$  is available, cf. (2.2). An object  $C$  of  $\mathcal{C}_Q$  is *faithful* if it is faithful as on object of  $\mathcal{C} = \mathcal{K}er(\mathcal{C}_Q)$  or, equivalently, if, with a slight abuse of the notation  $\vartheta_C$ , the induced homomorphism

$$\vartheta_C: U(\mathcal{C}) \longrightarrow \text{Aut}_{\mathcal{C}_Q}(C) \quad (2.5)$$

is injective.

The category  $\mathcal{C}_Q$  being a stably  $Q$ -graded symmetric monoidal category, the values of the functor  $k_Q$  now lie in the category of  $Q$ -monoids (monoids endowed with a  $Q$ -action that is compatible with the monoid structure). Given an object  $C$  of  $\mathcal{C}_Q$ , the grade homomorphism  $\text{Aut}_{\mathcal{C}_Q}(C) \rightarrow Q$  is surjective if and only if the class  $[C] \in k_Q \mathcal{C}_Q$  is fixed under  $Q$ . Hence the group  $\text{Aut}_{\mathcal{C}_Q}(C)$  of automorphisms in  $\mathcal{C}_Q$  of a faithful *invertible* object  $C$  of  $\mathcal{C}_Q$  whose class  $[C] \in k_Q \mathcal{C}_Q$  is fixed under  $Q$  fits into a group extension

$$e_C^{U(C)}: 1 \longrightarrow U(C) \xrightarrow{\vartheta_C} \text{Aut}_{\mathcal{C}_Q}(C) \longrightarrow Q \longrightarrow 1. \quad (2.6)$$

The category  $\mathcal{R}ep(Q, \mathcal{C}_Q)$  of representations of  $Q$  in  $\mathcal{C}_Q$  acquires an obvious symmetric monoidal category structure. In particular, the constituent  $E$  in the definition of a stably  $Q$ -graded symmetric monoidal category  $\mathcal{C}_Q$  is the unit object of  $\mathcal{R}ep(Q, \mathcal{C}_Q)$  [FW74, §3 p. 241], and  $U(\mathcal{R}ep(Q, \mathcal{C}_Q)) \cong H^0(Q, U(\mathcal{C}_Q))$  [FW74, (3.5) p. 242]. By definition, a member of  $\mathcal{R}ep(Q, \mathcal{C}_Q)$ , that is, a representation  $F: Q \rightarrow \text{Aut}_{\mathcal{C}_Q}(C)$  of  $Q$  by automorphisms in  $\mathcal{C}_Q$  of an object  $C$  of  $\mathcal{C}_Q$ , combined with the grade homomorphism  $\text{Aut}_{\mathcal{C}_Q}(C) \rightarrow Q$ , yields the identity map of  $Q$  whence the class  $[C] \in k_Q \mathcal{C}_Q$  is fixed under  $Q$ ; if furthermore,  $C$  is faithful, the representation  $F$  splits the associated group extension (2.6).

The notion of inverse and the definition of  $\mathcal{C}_Q$  being *group-like* extend to stably  $Q$ -graded symmetric monoidal categories in an obvious way. When the category  $\mathcal{C}_Q$  is group-like and has every object faithful, the assignment to an object  $C$  of  $\mathcal{C}_Q$  of the group extension (2.6) induces a homomorphism

$$\omega_{\mathcal{C}_Q}: H^0(Q, k_Q \mathcal{C}_Q) \longrightarrow H^2(Q, U(C)) \quad (2.7)$$

of abelian groups. By [FW74, Lemma 3.2 p. 241], if the given stably  $Q$ -graded symmetric monoidal category  $\mathcal{C}_Q$  is group-like, so is the category  $\mathcal{R}ep(Q, \mathcal{C}_Q)$ . In particular, in the proof of [FW74, Lemma 3.2 p. 241], an explicit construction is given for the inverse in the category  $\mathcal{R}ep(Q, \mathcal{C}_Q)$  associated to a group-like stably  $Q$ -graded symmetric monoidal category  $\mathcal{C}_Q$ .

The forgetful functor  $\mathcal{R}ep(Q, \mathcal{C}_Q) \rightarrow \mathcal{R}ep(\{e\}, \mathcal{C}_Q) \cong \mathcal{C}_Q$  induces a monoid homomorphism  $\mu_{\mathcal{C}_Q}: k\mathcal{R}ep(Q, \mathcal{C}_Q) \rightarrow k_Q \mathcal{C}_Q$  whose values lie in  $H^0(Q, k_Q \mathcal{C}_Q)$  since, given an object  $F$  of  $\mathcal{R}ep(Q, \mathcal{C}_Q)$ , that is, a representation  $F: Q \rightarrow \text{Aut}_{\mathcal{C}_Q}(C)$  of  $Q$  on an object  $C$  of  $\mathcal{C}_Q$ , the existence of the homomorphism  $F$  plainly entails that the grade homomorphism from  $\text{Aut}_{\mathcal{C}_Q}(C)$  to  $Q$  is surjective; in fact, when  $C$  is a faithful invertible object,  $F$  splits the group extension (2.6).

Let  $\mathcal{E}$  denote the full symmetric monoidal subcategory of  $\mathcal{C} = \mathcal{K}er(\mathcal{C}_Q)$  that has  $E(e)$  as its single object. This category is isomorphic to the abelian group  $\text{Aut}_{\mathcal{C}}(E(e)) \cong U(C)$ , viewed as a category with a single object. Let  $\mathcal{E}_Q$  denote the associated stably  $Q$ -graded symmetric monoidal category or, equivalently, the stably  $Q$ -graded symmetric monoidal subcategory of  $\mathcal{C}_Q$  having the single object  $E(e)$ ; this category is isomorphic to the group  $\text{Aut}_{\mathcal{C}_Q}(E(e)) \cong U(C) \rtimes Q$ , viewed as a category with a single object. The standard interpretation of  $H^1(Q, U(C))$  as classes of sections of (2.3), two sections being identified whenever they differ by conjugation in  $\text{Aut}_{\mathcal{C}_Q}(E(e))$  by a member of  $U(C)$ , induces a canonical isomorphism

$$H^1(Q, U(C)) \longrightarrow \mathcal{R}ep(Q, \mathcal{E}_Q) \quad (2.8)$$

of abelian monoids whence, since  $H^1(Q, U(C))$  is an abelian group, so is  $\mathcal{R}ep(Q, \mathcal{E}_Q)$  [FW74, (4.2) p. 243]. The obvious injection  $k\mathcal{R}ep(Q, \mathcal{E}_Q) \rightarrow k\mathcal{R}ep(Q, \mathcal{C}_Q)$  of abelian monoids yields an injection

$$j_{\mathcal{C}_Q}: H^1(Q, U(C)) \longrightarrow k\mathcal{R}ep(Q, \mathcal{C}_Q) \quad (2.9)$$

of abelian monoids. When  $C_Q$  is group-like and has all objects faithful, the sequence

$$0 \longrightarrow H^1(Q, U(C_Q)) \xrightarrow{j_{C_Q}} k\mathcal{R}ep(Q, C_Q) \xrightarrow{\mu_{C_Q}} H^0(Q, k_Q C_Q) \xrightarrow{\omega_{C_Q}} H^2(Q, U(C_Q)) \quad (2.10)$$

is an exact sequence of abelian groups [FW71b, Theorem 5 §6 p. 48], [FW74, Theorem 4.5 p. 244], [FW00, §1].

## 2.5 The stably $Q$ -graded Brauer category associated to a commutative ring endowed with a $Q$ -action

Given two Azumaya algebras  $A$  and  $B$ , a  $(B, A)$ -bimodule  $grade\ x \in Q$  is a  $(B, A)$ -bimodule  $M$  so that the left  $S$ -module structure of  $M$  via  $S \rightarrow B$  and the right  $S$ -module structure via  $S \rightarrow A$  are connected by the identity

$$ys = ({}^x s)y, \quad y \in M, \quad s \in S.$$

The  $Q$ -graded Brauer category  $\mathcal{B}_{S,Q}$  associated to the commutative ring  $S$  and the  $Q$ -action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  on  $S$  [FW71b, §3 p. 23], [FW74, p. 230], [FW00, §2], written in [FW71b, §3 p. 23] and [FW00, §2] as  $\mathcal{B}_R$  and in [FW74, p. 230] as  $\mathcal{B}r_R$ , has as *objects* the Azumaya  $S$ -algebras, a *morphism*  $([M], x): A \rightarrow B$  in  $\mathcal{B}_{S,Q}$  of *grade*  $x \in Q$  between two Azumaya algebras  $A$  and  $B$ , necessarily an isomorphism in  $\mathcal{B}_{S,Q}$ , being a pair  $([M], x)$  where  $[M]$  is an isomorphism class of an invertible  $(B, A)$ -bimodule  $M$  of grade  $x \in Q$ . Given three Azumaya algebras  $A, B, C$  and morphisms  $([{}_B M_A], x): A \rightarrow B$  and  $([{}_A M_C], x): C \rightarrow A$  in  $\mathcal{B}_{S,Q}$ , the composition of  $([{}_B M_A], x): A \rightarrow B$  with  $([{}_A M_C], x): C \rightarrow A$  in  $\mathcal{B}_{S,Q}$  is given by the morphism  $([{}_B M_A \otimes_A {}_A M_C], x): C \rightarrow B$  of grade  $x \in Q$  in  $\mathcal{B}_{S,Q}$ , where  $[{}_B M_A \otimes_A {}_A M_C]$  refers to the isomorphism class of the invertible  $(B, C)$ -bimodule  ${}_B M_A \otimes_A {}_A M_C$ . The operation of tensor product over the ground ring  $S$  and the assignment to an Azumaya  $S$ -algebra  $A$  of its opposite algebra  $A^{\text{op}}$  turn  $\mathcal{B}_{S,Q}$  into a group-like symmetric monoidal category [FW74, §5 p. 247], [FW00, §2] having  $(S, \kappa_Q: Q \rightarrow \text{Aut}(S))$  as unit object and  $U(\mathcal{B}_{S,Q}) = U(\mathcal{B}_S) = \text{Aut}_{\mathcal{B}_S}(S) = \text{Pic}(S)$  [FW00, §3], the assignment to an object  $A$  of  $\mathcal{B}_{S,Q}$  of the neutral element  $e$  of  $Q$  and that to a morphism in  $\mathcal{B}_{S,Q}$  of its grade in  $Q$  yields a functor  $g$  from  $\mathcal{B}_{S,Q}$  to  $Q$ , viewed as a category with a single object, i. e., turn  $\mathcal{B}_{S,Q}$  into a  $Q$ -graded symmetric monoidal category, and the grading is stable. In particular, the induced  $Q$ -action on  $U(\mathcal{B}_{S,Q}) = \text{Pic}(S)$  is the standard  $Q$ -action on  $\text{Pic}(S)$ . Since the category  $\mathcal{B}_{S,Q}$  is group-like, so is  $\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$ , and thence  $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$  is an abelian group. However, apart from trivial cases, this group is *not* the equivariant Brauer group of  $S$  relative to  $Q$ , this group being defined as the obvious equivariant generalization of the ordinary Brauer group and explored in Section 11 below.

By construction, then, the assignment to an automorphism in  $\mathcal{B}_{S,Q}$  of an Azumaya algebra  $A$  of its grade in  $Q$  yields a homomorphism

$$\pi^{\text{Aut}_{\mathcal{B}_{S,Q}}(A)}: \text{Aut}_{\mathcal{B}_{S,Q}}(A) \longrightarrow Q \quad (2.11)$$

which is surjective if and only if the Brauer class  $[A] \in B(S)$  of  $A$  in  $B(S)$  is fixed under  $Q$ , and the group  $\text{Aut}_{\mathcal{B}_{S,Q}}(A)$  associated to an Azumaya  $S$ -algebra  $A$  whose Brauer class  $[A]$  is fixed under  $Q$  fits into a group extension of the kind (2.6), viz.

$$e_A^{\text{Pic}(S)}: 1 \longrightarrow \text{Pic}(S) \longrightarrow \text{Aut}_{\mathcal{B}_{S,Q}}(A) \xrightarrow{\pi^{\text{Aut}_{\mathcal{B}_{S,Q}}(A)}} Q \longrightarrow 1 \quad (2.12)$$



with abelian kernel in such a way that the assignment to  $A$  of  $e_A^{\text{Pic}(S)}$  yields a homomorphism

$$\omega_{\mathcal{B}_{S,Q}} : H^0(Q, B(S)) \longrightarrow H^2(Q, \text{Pic}(S)). \quad (2.13)$$

The sequence (2.10) now takes the form

$$0 \longrightarrow H^1(Q, \text{Pic}(S)) \xrightarrow{j_{\mathcal{B}_{S,Q}}} k\mathcal{R}ep(Q, \mathcal{B}_{S,Q}) \xrightarrow{\mu_{\mathcal{B}_{S,Q}}} B(S)^Q \xrightarrow{\omega_{\mathcal{B}_{S,Q}}} H^2(Q, \text{Pic}(S)) \quad (2.14)$$

and is an exact sequence of abelian groups since the category  $\mathcal{B}_{S,Q}$  is group-like. This sequence is spelled out in [FW71b, Corollary 1 p. 51] as a sequence of abelian monoids, where the notation  $k\mathcal{R}ep\tilde{B}_S$  corresponds to our notation  $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$ . It is also a special case of the exact sequence of abelian groups given in [FW74, Corollary (4.6) p. 245]. The tilde notation in the setting of [FW71b] refers to the additional structure of an involution which, however, is not present in our approach.

The following is immediate; we spell it out for later reference.

**Proposition 2.1.** *Given two Azumaya  $S$ -algebras  $A$  and  $B$  and an invertible  $(B, A)$ -bimodule  $M$  so that the isomorphism class of  $M$  yields an isomorphism  $A \rightarrow B$  in  $\mathcal{B}_S$ , the induced isomorphism  $\text{Aut}_{\mathcal{B}_{S,Q}}(A) \rightarrow \text{Aut}_{\mathcal{B}_{S,Q}}(B)$  of groups is given by the assignment to an  $(A, A)$ -bimodule  $N_x$  of grade  $x \in Q$  of the  $(B, B)$ -bimodule*

$$M \otimes_A N_x \otimes_A M^* \cong \text{Hom}_A(M, M \otimes_A N_x)$$

of grade  $x$ . □

## 2.6 Picard categories

The *Picard category*  $\mathcal{P}ic_S$  associated to the commutative ring  $S$ , written in [FW71b, §2 p. 17], [FW00, §2] as  $\mathcal{C}_R$ , has as objects the faithful finitely generated invertible projective  $S$ -modules, that is, the faithful finitely generated projective rank one  $S$ -modules, a morphism in  $\mathcal{P}ic_S$  an isomorphism between two  $S$ -modules in  $\mathcal{P}ic_S$ . Given three faithful finitely generated projective rank one  $S$ -modules and two morphisms between them, composition in  $\mathcal{P}ic_S$  is defined in the obvious way, that is, via ordinary composition of  $S$ -linear maps. The operation of tensor product over the ground ring  $S$  and the assignment to a faithful finitely generated projective rank one  $S$ -module of its  $S$ -dual turn  $\mathcal{P}ic_S$  into a group-like symmetric monoidal category having the ground ring  $S$ , viewed as a free rank one  $S$ -module, as its unit object and  $U(\mathcal{P}ic_S) = \text{Aut}_{\mathcal{P}ic_S}(S) = U(S)$ , the group of units of the ground ring  $S$ . The abelian group  $k\mathcal{P}ic_S$  is canonically isomorphic to the ordinary Picard group  $\text{Pic}(S)$  of  $S$ .

The  *$Q$ -graded Picard category*  $\mathcal{P}ic_{S,Q}$  associated to the commutative ring  $S$  and the  $Q$ -action  $\kappa_Q : Q \rightarrow \text{Aut}(S)$  on  $S$ , written in [FW00, §3] as  $\mathcal{C}_R$ , has the same objects as  $\mathcal{P}ic_S$ , a morphism  $(f, x) : J_1 \rightarrow J_2$  in  $\mathcal{P}ic_{S,Q}$  of grade  $x \in Q$  between two faithful projective rank one  $S$ -modules  $J_1$  and  $J_2$ , necessarily an isomorphism in  $\mathcal{P}ic_{S,Q}$ , being a pair  $(f, x)$  where  $f : J_1 \rightarrow J_2$  is an isomorphism over  $R = S^Q$  such that  $f(sy) = {}_x s f(y)$ , for  $s \in S$  and  $y \in J_1$ . Given three faithful finitely generated projective rank one  $S$ -modules and two morphisms between them, composition in  $\mathcal{P}ic_{S,Q}$  is defined in the obvious way, that is, via ordinary composition of  $S$ -linear maps. The operation of tensor product over the ground ring  $S$  and the assignment to a faithful finitely generated projective rank one  $S$ -module of its  $S$ -dual turn  $\mathcal{P}ic_{S,Q}$  into a group-like symmetric monoidal category having  $(S, \kappa_Q : Q \rightarrow \text{Aut}(S))$  as unit object and

$U(\mathcal{P}ic_{S,Q}) = U(\mathcal{P}ic_S) = \text{Aut}_{\mathcal{P}ic_S}(S) = U(S)$ , the group of units of the ground ring  $S$ . The assignment to an object of  $\mathcal{P}ic_{S,Q}$  of the neutral element  $e$  of  $Q$  and that to a morphism of its grade in  $Q$  turn  $\mathcal{P}ic_{S,Q}$  into a  $Q$ -graded symmetric monoidal category, and the grading is stable. In particular, the induced  $Q$ -action on  $U(\mathcal{P}ic_{S,Q}) = U(S)$  is the standard  $Q$ -action on  $U(S)$ , endowed with its induced  $Q$ -module structure. Since the category  $\mathcal{P}ic_{S,Q}$  is group-like, so is the category  $\mathcal{R}ep(Q, \mathcal{P}ic_{S,Q})$ , and thence  $k\mathcal{R}ep(Q, \mathcal{P}ic_{S,Q})$  is an abelian group. This group is canonically isomorphic to the equivariant Picard group  $\text{EPic}(S, Q)$  of  $S$  with respect to the  $Q$ -action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  [FW71b, §5 p. 38], [FW71a], [FW00, §3] (written as  $C(R, \Gamma)$  and referred to as the equivariant class group). The sequence (2.10) now takes the form

$$0 \longrightarrow H^1(Q, U(S)) \xrightarrow{j_{\mathcal{P}ic_{S,Q}}} \text{EPic}(S, Q) \xrightarrow{\mu_{\mathcal{P}ic_{S,Q}}} \text{Pic}(S)^Q \xrightarrow{\omega_{\mathcal{P}ic_{S,Q}}} H^2(Q, U(S)) \quad (2.15)$$

and is an exact sequence of abelian groups since the category  $\mathcal{P}ic_{S,Q}$  is group-like. This construction recovers the classical four-term exact sequence associated to the data; this sequence can, of course, be obtained by straightforward ad hoc constructions. For example, the  $Q$ -action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  splits the group extension

$$e_{\mathcal{P}ic_{S,Q}}: 0 \longrightarrow U(S) \longrightarrow \text{Aut}_{\mathcal{P}ic_{S,Q}}(S) \xrightarrow{\pi^{\text{Aut}_{\mathcal{P}ic_{S,Q}}(A)}} Q \longrightarrow 1, \quad (2.16)$$

and the homomorphism  $j_{\mathcal{P}ic_{S,Q}}$  is induced by the assignment to a derivation  $d: Q \rightarrow U(S)$  of the associated section for  $\pi^{\text{Aut}_{\mathcal{P}ic_{S,Q}}(A)}$ . For later reference we note that the exactness of (2.15) at  $\text{EPic}(S, Q)$  says that

$$j_{\mathcal{P}ic_{S,Q}}: H^1(Q, U(S)) \longrightarrow \text{EPic}(S|S, Q) \quad (2.17)$$

is an isomorphism from  $H^1(Q, U(S))$  onto the subgroup  $\text{EPic}(S|S, Q)$  of  $\text{EPic}(S, Q)$  which consists of classes of objects in  $\mathcal{R}ep(Q, \mathcal{P}ic_{S,Q})$  whose underlying  $S$ -modules are free of rank 1.

## 2.7 Change of actions

We define the *change of actions category*  $\mathcal{C}hange$  as follows: The *objects* of  $\mathcal{C}hange$  are triples  $(S, Q, \kappa)$  that consist of a commutative ring  $S$ , a group  $Q$ , and an action  $\kappa: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$ ; given two objects  $(S, Q, \kappa_Q)$  and  $(T, G, \kappa_G)$ , a *morphism*  $(f, \varphi): (S, Q, \kappa) \longrightarrow (T, G, \kappa_G)$  in  $\mathcal{C}hange$  consists of a ring homomorphism  $f: S \rightarrow T$  and a group homomorphism  $\varphi: Q \rightarrow G$  such that, given  $s \in S$  and  $x \in Q$ ,

$$f(\varphi(x)s) = x(f(s)). \quad (2.18)$$

In this category, composition of morphisms is defined in the obvious way.

To describe the morphisms in a somewhat more picturesque way, given the ring homomorphism  $f: S \rightarrow T$ , let  $\text{Aut}^S(T)$  denote the subgroup of  $\text{Aut}(S) \times \text{Aut}(T)$  that consists of those pairs  $(\alpha, \beta)$  of automorphisms which have the property that  $\beta \circ f = f \circ \alpha$ , and let  $\text{Aut}(T|S)$  denote the kernel of the obvious homomorphism  $\text{Aut}^S(T) \rightarrow \text{Aut}(S)$ . In the special case where  $f$  is injective, the group  $\text{Aut}(T|S)$  amounts to the ordinary group of automorphisms of  $T$  that leave  $S$  elementwise fixed whence the notation. In the general case, given, furthermore,

the homomorphism  $\varphi: G \rightarrow Q$ , the condition (2.18) says that  $(\kappa_Q \circ \varphi, \kappa_G): G \rightarrow \text{Aut}^S(T)$  yields a commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & \ker(\varphi) & \longrightarrow & G & \xrightarrow{\varphi} & Q \\
& & \downarrow & & (\kappa_Q \circ \varphi, \kappa_G) \downarrow & & \kappa_Q \downarrow \\
1 & \longrightarrow & \text{Aut}(T|S) & \longrightarrow & \text{Aut}^S(T) & \longrightarrow & \text{Aut}(S)
\end{array} \tag{2.19}$$

in the category of groups with exact rows.

The stably  $Q$ -graded categories  $\mathcal{B}_{S,Q}$  and  $\mathcal{Pic}_{S,Q}$  behave functorially on *Change* in an obvious way with respect to the variable  $(S, Q)$ . Likewise, group cohomology  $H^*(Q, \text{U}(S))$  is a covariant functor on the change of actions category *Change* with respect to the variable  $(S, Q)$ , and so are  $k\mathcal{Rep}(Q, \mathcal{B}_{S,Q})$  and  $\text{EPic}(S, Q)$ . More examples will show up later in the paper.

### 3 Normal algebras and their Teichmüller complexes

#### 3.1 Normal algebras

Let  $A$  be a central  $S$ -algebra. Denote by  $\text{Aut}(A)$  the group of ring automorphisms of  $A$  and by  $\text{U}(A)$  the group of units of  $A$ . The obvious homomorphism  $\partial: \text{U}(A) \rightarrow \text{Aut}(A)$  assigns to a unit of  $A$  the associated inner automorphism of  $A$ , and the obvious action of  $\text{Aut}(A)$  on  $\text{U}(A)$  turns the triple  $(\text{U}(A), \text{Aut}(A), \partial)$  into a crossed module. Moreover,  $\ker(\partial) = \text{U}(S)$ , the group of units of  $S$ , and  $\partial(\text{U}(A))$  is a normal subgroup of  $\text{Aut}(A)$ . As in the introduction, write  $\text{Out}(A) = \text{coker}(\partial)$ .

Each inner automorphism of  $A$  leaves  $S$  elementwise fixed whence the restriction map  $\text{Aut}(A) \rightarrow \text{Aut}(S)$  induces a homomorphism  $\text{Out}(A) \rightarrow \text{Aut}(S)$ , and the data fit into the crossed 2-fold extension

$$e_A: 0 \longrightarrow \text{U}(S) \longrightarrow \text{U}(A) \xrightarrow{\partial} \text{Aut}(A) \longrightarrow \text{Out}(A) \longrightarrow 1. \tag{3.1}$$

As in the introduction, let  $Q$  be a group and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of  $Q$  on  $S$  by ring automorphisms. With respect to  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ , let

$$\text{Aut}(A, Q) = \text{Aut}(A) \times_{\text{Aut}(S)} Q, \quad \text{Out}(A, Q) = \text{Out}(A) \times_{\text{Aut}(S)} Q$$

denote the indicated fiber product groups. The group  $\text{Aut}(A, Q)$  acts on  $\text{U}(A)$  in the obvious way, and this action, together with the obvious map  $\partial: \text{U}(A) \rightarrow \text{Aut}(A, Q)$ , yields a crossed 2-fold extension

$$e_{(A,Q)}: 0 \longrightarrow \text{U}(S) \longrightarrow \text{U}(A) \xrightarrow{\partial} \text{Aut}(A, Q) \longrightarrow \text{Out}(A, Q) \longrightarrow 1. \tag{3.2}$$

As in the introduction, we define a  $Q$ -normal structure on the central  $S$ -algebra  $A$  relative to the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$  to be a homomorphism  $\sigma: Q \rightarrow \text{Out}(A)$  that lifts the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$  in the sense that the composite of  $\sigma$  with the obvious restriction map  $\text{res}: \text{Out}(A) \rightarrow \text{Aut}(S)$  coincides with  $\kappa_Q$ . A  $Q$ -normal algebra is, then, a central  $S$ -algebra  $A$  together with a  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(A)$ .

We shall occasionally refer to the canonical homomorphism  $\text{Aut}(A, Q) \rightarrow Q$  as well as to the canonical homomorphism  $\text{Out}(A, Q) \rightarrow Q$  as a *grade homomorphism* and to the value in  $Q$  of a member of  $\text{Aut}(A, Q)$  and, likewise, of a member of  $\text{Out}(A, Q)$ , as the *grade* of that member.

### 3.2 Discussion of normality and generalized normality

Given a central  $S$ -algebra  $A$ , we will say that  $A$  is *weakly  $Q$ -normal* if each automorphism  $\kappa_Q(x)$  of  $S$ , as  $x$  ranges over  $Q$ , extends to a ring automorphism of  $A$ . For Azumaya algebras, this is the notion of  $Q$ -normality used by Childs [Chi72], Pareigis [Par64], Ulbrich [Ul89], [Ul94], and Zelinski [Zel76]. Equivalently, a central  $S$ -algebra  $A$  is weakly  $Q$ -normal if and only if the canonical homomorphism  $\pi^{\text{Out}(A, Q)}: \text{Out}(A, Q) \rightarrow Q$  is surjective.

Let  $A$  be an Azumaya  $S$ -algebra and let  $\lambda$  be automorphisms of  $A$  over the center  $S$  of  $A$ . Let  ${}_{\lambda}A$  be the  $(A \otimes A^{\text{op}})$ -module which, as an  $S$ -module, is just  $A$ , and whose structure map is given by

$$(A \otimes A^{\text{op}}) \otimes {}_{\lambda}A \longrightarrow {}_{\lambda}A, (a \otimes b) \otimes y \longmapsto \lambda(a)yb, \quad a, b, y \in A.$$

Then

$${}_{\lambda}J = \text{Hom}_{A^e}(A, {}_{\lambda}A) \cong \{a \in A; \lambda(x)a = ax, \text{ for all } x \in A\}$$

is a faithful projective rank one  $S$ -module in such a way that the canonical evaluation map

$$\text{Hom}_{A^e}(A, {}_{\lambda}A) \otimes A \rightarrow {}_{\lambda}A$$

is an isomorphism of  $A^e$ -modules, the  $A^e$ -module structure on the left-hand side being the one induced by the canonical  $A^e$ -module structure on  $A$ .

Let  $\text{Pic}(A)$  denote the abelian group (under the operation of taking tensor products) that consists of left  $A$ -isomorphism classes of left  $(A \otimes A^{\text{op}})$ -modules  $P$  which, as  $S$ -modules, are finitely generated and projective, such that, for every maximal ideal  $\mathfrak{m}$  in  $S$ , the  $S_{\mathfrak{m}}$ -module  $P \otimes S_{\mathfrak{m}}$  is isomorphic to  $A \otimes S_{\mathfrak{m}}$ .

Recall the following generalization of the Skolem-Noether theorem [RZ61, Theorem 5].

**Proposition 3.1.** *Given an Azumaya  $S$ -algebra  $A$ , the assignment to an automorphism  $\lambda$  of  $A$  over  $S$  of the class  $\alpha(\lambda) = [{}_{\lambda}J] \in \text{Pic}(S)$  and the assignment to the class  $[J] \in \text{Pic}(S)$  of the class  $\beta([J]) = [A \otimes J] \in \text{Pic}(A)$  yields an exact sequence*

$$1 \longrightarrow \text{Out}(A|S) \xrightarrow{\alpha} \text{Pic}(S) \xrightarrow{\beta} \text{Pic}(A) \longrightarrow 1$$

*of abelian groups. In particular, the group  $\text{Out}(A|S)$  is an abelian group, and the image  $\alpha(\text{Out}(A|S)) \subseteq \text{Pic}(S)$  consists of isomorphism classes of projective rank one modules  $J$  such that  $J \otimes A$  is isomorphic to  $A$  as a left  $A$ -module.*

The following is immediate.

**Corollary 3.2.** *A weakly  $Q$ -normal Azumaya algebra  $A$  admits a  $Q$ -normal structure if and only if the associated group extension*

$$e_A^{\text{Out}(A|S)}: 1 \longrightarrow \text{Out}(A|S) \longrightarrow \text{Out}(A, Q) \xrightarrow{\pi^{\text{Out}(A, Q)}} Q \longrightarrow 1 \quad (3.3)$$

*(with abelian kernel) splits, and  $Q$ -normal structures on  $A$  are then in one-one correspondence with sections  $\sigma: Q \rightarrow \text{Out}(A, Q)$  for  $\pi^{\text{Out}(A, Q)}$ .*

The classes of weakly  $Q$ -normal Azumaya  $S$ -algebras in  $B(S)$  constitute a subgroup  $B(S, Q) \subseteq B(S)^Q$  of the subgroup  $B(S)^Q$  of  $B(S)$  that consists of the  $Q$ -invariant Brauer classes. When the class  $[A] \in B(S)$  of an Azumaya  $S$ -algebra  $A$  is fixed under  $Q$ , the algebra  $A$  need not be weakly  $Q$ -normal, however.

Given a ring automorphism  $f: B \rightarrow B$  of an  $S$ -algebra  $B$ , we will denote by  $B_f$  the  $(B, B)$ -bimodule which, as a left  $S$ -module, is just  $B$  and whose structure map is given by

$$B \otimes B_f \otimes B \longrightarrow B_f, \quad b_1 \cdot b \cdot b_2 = b_1 b f(b_2), \quad b, b_1, b_2 \in B.$$

Given an Azumaya  $S$ -algebra  $A$ , the assignment to an automorphism  $\alpha_x$  of  $A$  that extends the automorphism  $\kappa_Q(x)$  of  $S$ , as  $x$  ranges over the image  $\pi^{\text{Out}(A, Q)}(\text{Out}(A, Q)) \subseteq Q$  of  $\text{Out}(A, Q)$  in  $Q$  under  $\pi^{\text{Out}(A, Q)}$ , of the invertible  $(A, A)$ -bimodule  $A_{\alpha_x}$  induces an injective homomorphism

$$\Theta: \text{Out}(A, Q) \longrightarrow \text{Aut}_{\mathcal{B}_{S, Q}}(A) \quad (3.4)$$

such that the composite

$$\text{Out}(A, Q) \xrightarrow{\Theta} \text{Aut}_{\mathcal{B}_{S, Q}}(A) \xrightarrow{\pi^{\text{Aut}_{\mathcal{B}_{S, Q}}(A)}} Q \quad (3.5)$$

coincides with  $\pi^{\text{Out}(A, Q)}: \text{Out}(A, Q) \rightarrow Q$ . Hence:

**Proposition 3.3.** *An Azumaya  $S$ -algebra  $A$  whose Brauer class  $[A] \in \text{B}(S)$  in  $\text{B}(S)$  is fixed under  $Q$  is weakly  $Q$ -normal if and only if the composite (3.5) is surjective.  $\square$*

Thus, given a weakly  $Q$ -normal Azumaya  $S$ -algebra  $A$ , the diagram

$$\begin{array}{ccccccc} e_A^{\text{Out}(A|S)}: 1 & \longrightarrow & \text{Out}(A|S) & \longrightarrow & \text{Out}(A, Q) & \xrightarrow{\pi^{\text{Out}(A, Q)}} & Q \longrightarrow 1 \\ & & \alpha \downarrow & & \Theta \downarrow & & \parallel \\ e_A^{\text{Pic}(S)}: 1 & \longrightarrow & \text{Pic}(S) & \longrightarrow & \text{Aut}_{\mathcal{B}_{S, Q}}(A) & \xrightarrow{\pi^{\text{Aut}_{\mathcal{B}_{S, Q}}(A)}} & Q \longrightarrow 1 \end{array} \quad (3.6)$$

is commutative with exact rows.

*Remark 3.4.* In particular, an Azumaya algebra  $A$  having  $\alpha: \text{Out}(A|S) \rightarrow \text{Pic}(S)$  surjective is weakly  $Q$ -normal if its Brauer class  $[A] \in \text{B}(S)$  is fixed under  $Q$ . When  $S$  is a field,  $\text{Pic}(S)$  is trivial whence then an Azumaya algebra  $A$  is weakly  $Q$ -normal, in fact even normal, if and only if its Brauer class  $[A] \in \text{B}(S)$  is fixed under  $Q$ .

*Remark 3.5.* Given a weakly  $Q$ -normal Azumaya  $S$ -algebra  $A$ , there is a fundamental difference between the two group extensions  $e_A^{\text{Out}(A|S)}$  and  $e_A^{\text{Pic}(S)}$ : The kernel of the former depends on  $A$  whereas that of the latter does not.

A  $Q$ -normal Azumaya  $S$ -algebra  $(A, \sigma)$  represents a member of  $\text{B}(S, Q)(\subseteq \text{B}(S)^Q)$  in such a way that  $\sigma$  splits the associated group extension (3.3), and the composite

$$\Theta_\sigma: Q \xrightarrow{\sigma} \text{Out}(A, Q) \xrightarrow{\Theta} \text{Aut}_{\mathcal{B}_{S, Q}}(A) \quad (3.7)$$

of  $\sigma$  with the homomorphism  $\Theta$  from  $\text{Out}(A, Q)$  to  $\text{Aut}_{\mathcal{B}_{S, Q}}(A)$  given above as (3.4) yields the object  $(A, \Theta_\sigma)$  of  $\mathcal{R}ep(Q, \mathcal{B}_{S, Q})$  and hence a member of  $k\mathcal{R}ep(Q, \mathcal{B}_{S, Q})$ . We shall therefore refer to an object of  $\mathcal{R}ep(Q, \mathcal{B}_{S, Q})$  as a *generalized  $Q$ -normal Azumaya algebra*. In Subsection 3.10 we shall show that, when the group  $Q$  is finite, each member of  $k\mathcal{R}ep(Q, \mathcal{B}_{S, Q})$  arises in this manner from a  $Q$ -normal Azumaya  $S$ -algebra.

### 3.3 Equivariant algebras and scalar extension

Let  $Q$  be a group and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of  $Q$  on  $S$ . A  $Q$ -equivariant  $S$ -algebra  $(A, \tau)$  consists of a central  $S$ -algebra  $A$  together with a homomorphism  $\tau: Q \rightarrow \text{Aut}(A)$  that induces  $\kappa_Q$ .

Let  $R = S^Q$  be the subring of  $S$  which is elementwise fixed under the  $Q$ -action. Given a central  $R$ -algebra  $B$ , *scalar extension*  $B \mapsto A = B \otimes_R S$  yields the central  $S$ -algebra  $A$ ; then the action of  $Q$  on  $A = B \otimes_R S$  induced by the action  $\kappa_Q$  of  $Q$  on  $S$  yields a  $Q$ -equivariant structure  $\tau_0: Q \rightarrow \text{Aut}(A)$  and hence a  $Q$ -normal structure  $\sigma_0: Q \rightarrow \text{Out}(A)$ , and we will say that  $(A, \tau_0)$  and  $(A, \sigma_0)$  arise from  $B$  by *scalar extension*. By Galois descent (1.2) (ii), if  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ , any  $Q$ -equivariant  $S$ -algebra arises by scalar extension.

### 3.4 The Teichmüller class of a $Q$ -normal algebra

As in the classical approach of Teichmüller [Tei40] and Eilenberg-Mac Lane [EM48], we seek to classify  $Q$ -normal  $S$ -algebras modulo those which are obtained by extension of scalars, rather than modulo the equivariant ones (in case of algebras over fields this makes no difference by Galois descent), by means of certain 3-dimensional cohomology classes.

Let  $(A, \sigma)$  be a  $Q$ -normal  $S$ -algebra, with respect to  $\sigma: Q \rightarrow \text{Out}(A)$ , let  $B^\sigma$  denote the fiber product group  $\text{Aut}(A) \times_{\text{Out}(A)} Q$ , let  $B^\sigma$  act on  $U(A)$  in the obvious way, that is, via the canonical homomorphism from  $B^\sigma$  to  $\text{Aut}(A)$ , and let  $\partial^\sigma: U(A) \rightarrow B^\sigma$  denote the homomorphism induced by  $\partial: U(A) \rightarrow \text{Aut}(A)$ . Pulling back the crossed 2-fold extension (3.1) above yields the crossed 2-fold extension

$$e_{(A, \sigma)}: 0 \longrightarrow U(S) \longrightarrow U(A) \xrightarrow{\partial^\sigma} B^\sigma \longrightarrow Q \longrightarrow 1, \quad (3.8)$$

uniquely determined by  $(A, \sigma)$ . By construction,  $B^\sigma$  may be identified with a certain subgroup of  $\text{Aut}(A, Q)$ . The crossed 2-fold extension  $e_{(A, \sigma)}$  represents a class  $[e_{(A, \sigma)}] \in H^3(Q, U(S))$  [Hue80, Section 7], cf. Section 1 above. We shall refer to  $e_{(A, \sigma)}$  as the *Teichmüller complex* of  $(A, \sigma)$  and to the corresponding class in  $H^3(Q, U(S))$  as the *Teichmüller class* of  $(A, \sigma)$ .

For completeness, and for future reference, we indicate how the Teichmüller complex is related with the classical “Teichmüller cocycle”: Let  $e_Q$  denote the crossed standard resolution of  $Q$  introduced in [Hue80, Section 9], and lift the identity map to a commutative diagram of the kind

$$\begin{array}{ccccccccc} e_Q: & \dots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 1 \\ & & & \downarrow \xi & & \downarrow \beta & & \downarrow \alpha & & \parallel & & \\ e_{(A, \sigma)}: & 0 & \longrightarrow & U(S) & \longrightarrow & U(A) & \longrightarrow & B^\sigma & \longrightarrow & Q & \longrightarrow & 1, \end{array}$$

so that  $(\beta, \alpha)$  is a morphism of crossed modules, cf. [Hue80, Section 5]. In view of [Hue80, Section 9 (\*\*)], the map  $\xi$  is a 3-cocycle of  $Q$  with values in  $U(S)$ . By the main Theorem in [Hue80], the class  $[e_{(A, \sigma)}] \in H^3(Q, U(S))$  coincides with the class which, in the cocycle description, is represented by  $\xi$ ; more precisely, the isomorphism  $\text{Opext}^2(Q, U(S)) \rightarrow H^3(Q, U(S))$  results from the assignment to a crossed 2-fold extension of a 3-cocycle by the method just explained. It is manifest that, in case  $S$  is a field,  $Q$  a finite group of automorphisms of  $S$  and  $A$  a finite dimensional central simple  $S$ -algebra, the 3-cocycle  $\xi$  is a Teichmüller cocycle for

$A$ , cf. Teichmüller [Tei40] and Eilenberg and Mac Lane [EM48], and so  $[e_{(A,\sigma)}]$  then comes down to the Teichmüller class represented by the Teichmüller cocycle.

Henceforth we will denote by  $e_0$  the crossed 2-fold extension

$$e_0: 0 \longrightarrow U(S) \xrightarrow{=} U(S) \xrightarrow{0} Q \xrightarrow{=} Q \longrightarrow 1,$$

the crossed module structure being given by the  $Q$ -module structure on  $U(S)$ .

**Proposition 3.6.** *The Teichmüller class  $[e_{(A,\sigma)}]$  of an equivariant  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  is zero.*

*Proof.* There is a congruence morphism  $(1, \cdot, \cdot, 1): e_0 \rightarrow e_{(A,\sigma)}$  of crossed 2-fold extensions, and  $[e_0] = 0 \in H^3(Q, U(S))$ , see [Hue80].  $\square$

### 3.5 Opposite algebras

Given an algebra  $A$ , we denote its opposite algebra by  $A^{\text{op}}$  as usual. Let  $A$  be a central  $S$ -algebra. The association

$$\alpha \mapsto \hat{\alpha}, \quad \hat{\alpha}(a^{\text{op}}) = (\alpha a)^{\text{op}}, \quad a \in A, \alpha \in \text{Aut}(A),$$

yields an isomorphism  $\hat{\cdot}: \text{Aut}(A) \rightarrow \text{Aut}(A^{\text{op}})$  and, with an abuse of notation, we denote by  $\hat{\cdot}: \text{Out}(A) \rightarrow \text{Out}(A^{\text{op}})$  the induced isomorphism as well. Moreover, inversion yields an isomorphism  $(^{\text{op}})^{-1}: U(A) \rightarrow U(A^{\text{op}})$ , and so we get an isomorphism

$$(-1, (^{\text{op}})^{-1}, \hat{\cdot}, \hat{\cdot}): e_A \longrightarrow e_{A^{\text{op}}}$$

of crossed 2-fold extensions.

Given a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$ , we equip  $A^{\text{op}}$  with a  $Q$ -normal structure by setting  $\sigma^{\text{op}} = \hat{\cdot} \circ \sigma$ ; we refer to  $(A^{\text{op}}, \sigma^{\text{op}})$  as the *opposite* of  $(A, \sigma)$ . Likewise, given a  $Q$ -equivariant  $S$ -algebra  $(A, \tau)$ , letting  $\tau^{\text{op}} = \hat{\cdot} \circ \tau$ , we obtain a  $Q$ -equivariant structure on  $A^{\text{op}}$ , and we refer to  $(A^{\text{op}}, \tau^{\text{op}})$  as the *opposite* of  $(A, \tau)$ .

**Proposition 3.7.** *For a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$ , the isomorphism  $\hat{\cdot}$  induces an isomorphism*

$$(-1, \cdot, \cdot, 1): e_{(A,\sigma)} \longrightarrow e_{(A^{\text{op}}, \sigma^{\text{op}})}$$

*of crossed 2-fold extensions, and therefore*

$$[e_{(A,\sigma)}] + [e_{(A^{\text{op}}, \sigma^{\text{op}})}] = 0 \in H^3(Q, U(S)).$$

### 3.6 Matrix algebras

Now let  $A$  be a central  $S$ -algebra, and let  $M_I(A)$  be a matrix algebra over  $A$ ; if  $I$  is not finite, we interpret  $M_I(A)$  as being the endomorphism ring of  $\oplus_I A^{\text{op}}$ . The algebra  $M_I(A)$  is again a central  $S$ -algebra. It is obvious that an automorphism of  $A$  yields one of  $M_I(A)$  in a unique way, and the obvious map  $A \rightarrow M_I(A)$  is a ring homomorphism. Hence a  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(A)$  on  $A$  determines one on  $M_I(A)$ , and we denote this structure by  $\sigma_I: Q \rightarrow \text{Out}(M_I(A))$ ; likewise a  $Q$ -equivariant structure  $\tau: Q \rightarrow \text{Aut}(A)$  on  $A$  determines an obvious  $Q$ -equivariant structure on  $M_I(A)$ , which we shall denote by  $\tau_I: Q \rightarrow \text{Aut}(M_I(A))$ . A special case is  $A = S$  and  $\sigma = \tau = \kappa_Q$ ; then we get the obvious equivariant structure  $\kappa_{Q,I}$  on  $M_I(S)$ .

**Proposition 3.8.** *Given a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$ , the obvious algebra map  $A \rightarrow M_I(A)$  extends to a morphism  $(A, \sigma) \rightarrow (M_I(A), \sigma_I)$  of  $Q$ -normal  $S$ -algebras, and there is an induced congruence  $(1, \cdot, \cdot, 1): e_{(A, \sigma)} \rightarrow e_{(M_I(A), \sigma_I)}$  involving the corresponding crossed 2-fold extensions. Hence*

$$[e_{(A, \sigma)}] = [e_{(M_I(A), \sigma_I)}] \in H^3(Q, U(S)).$$

### 3.7 Tensor products

Given two  $Q$ -normal  $S$ -algebras  $(A_1, \sigma_1)$  and  $(A_2, \sigma_2)$ , the  $Q$ -normal structures determine an obvious homomorphism  $\sigma_1 \otimes \sigma_2: Q \rightarrow \text{Out}(A_1 \otimes A_2)$  so that  $(A_1 \otimes A_2, \sigma_1 \otimes \sigma_2)$  is a  $Q$ -normal  $S$ -algebra; likewise, given two  $Q$ -equivariant  $S$ -algebras  $(A_1, \tau_1)$  and  $(A_2, \tau_2)$ , the  $Q$ -equivariant structures determine an obvious homomorphism  $\tau_1 \otimes \tau_2: Q \rightarrow \text{Aut}(A_1 \otimes A_2)$  in such a way that  $(A_1 \otimes A_2, \tau_1 \otimes \tau_2)$  is a  $Q$ -equivariant  $S$ -algebra.

Given two  $Q$ -normal  $S$ -algebras  $(A_1, \sigma_1)$  and  $(A_2, \sigma_2)$ , consider the Baer sum

$$e_{(A_1, \sigma_1)} + e_{(A_2, \sigma_2)}: 0 \rightarrow U(S) \rightarrow U(A_1) \times^{U(S)} U(A_2) \rightarrow B^{\sigma_1} \times_Q B^{\sigma_2} \rightarrow Q \rightarrow 1$$

of the crossed 2-fold extensions  $e_{(A_1, \sigma_1)}$  and  $e_{(A_2, \sigma_2)}$ , cf. [Hue80] for details. This is a crossed 2-fold extension that represents the sum  $[e_{(A_1, \sigma_1)}] + [e_{(A_2, \sigma_2)}]$  in  $H^3(Q, U(S))$ .

**Proposition 3.9.** *There is an obvious congruence*

$$(1, \cdot, \cdot, 1): e_{(A_1, \sigma_1)} + e_{(A_2, \sigma_2)} \rightarrow e_{(A_1 \otimes A_2, \sigma_1 \otimes \sigma_2)}$$

*of crossed 2-fold extensions. Hence*

$$[e_{(A_1 \otimes A_2, \sigma_1 \otimes \sigma_2)}] = [e_{(A_1, \sigma_1)}] + [e_{(A_2, \sigma_2)}] \in H^3(Q, U(S)). \quad \square$$

Combining Propositions 3.7 and 3.9 with the observation that the Teichmüller class of  $(S, \kappa_Q)$  is zero, we see that the Teichmüller classes of  $Q$ -normal  $S$ -algebras constitute a subgroup of  $H^3(Q, U(S))$ .

### 3.8 Behaviour under change of actions

**Proposition 3.10.** *Let  $(f, \varphi): (S, Q, \kappa) \rightarrow (T, G, \lambda)$  be a morphism in **Change** between two given objects  $(S, Q, \kappa)$  and  $(T, G, \lambda)$  of **Change**, so that the group  $G$  acts on  $S$  via  $\varphi: G \rightarrow Q$ .*

(i) *Given a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$ , the structure map  $\sigma$  and  $(f, \varphi)$  induce a canonical  $G$ -normal structure  $\sigma_{(f, \varphi)}: G \rightarrow \text{Out}(T \otimes A)$  on  $T \otimes A$  that is compatible with the operations of taking opposite algebras and tensor products.*

(ii) *Given a  $Q$ -equivariant  $S$ -algebra  $(A, \tau)$ , the structure map  $\tau$  and  $(f, \varphi)$  induce a canonical  $G$ -equivariant structure  $\tau_{(f, \varphi)}: G \rightarrow \text{Aut}(T \otimes A)$  on  $T \otimes A$  that is compatible with the operations of taking opposite algebras and tensor products.*

(iii) *Given a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$ , the structure map  $\sigma$  and  $(f, \varphi)$  induce, in a canonical way, morphisms*

$$(1, \cdot, \cdot, \varphi): e_{(A, \sigma \varphi)} \rightarrow e_{(A, \sigma)} \text{ and } (f, \cdot, \cdot, 1): e_{(A, \sigma \varphi)} \rightarrow e_{(T \otimes A, \sigma_{(f, \varphi)})}$$

*of crossed 2-fold extensions. Consequently,*

$$[e_{(T \otimes A, \sigma_{(f, \varphi)})}] = (f, \varphi)_*[e_{(A, \sigma)}] \in H^3(G, U(T)),$$

*where  $(f, \varphi)_*$  denotes the map induced on cohomology.*  $\square$



### 3.9 Embedding algebras into algebras with smaller center

The title of this subsection is intended to remind the reader of Deuring's paper [Deu36] having the same title.

As before,  $Q$  denotes a group and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of  $Q$  on the commutative ring  $S$ . Let  $A$  be a central  $S$ -algebra, let  $R$  denote the subring  $S^Q$  of elements of  $S$  that are elementwise fixed under  $Q$ , and let  $C$  be an  $R$ -algebra containing  $A$  as a subalgebra. We shall refer to the embedding of  $A$  into  $C$  as a *Deuring embedding with respect to the action*  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $A$  if each automorphism  $\kappa_Q(x)$ , as  $x$  ranges over  $Q$ , extends to an inner automorphism of  $C$  that maps  $A$  to itself. In the special case where  $A$  coincides with the centralizer of  $T$  in  $C$ , the requirement that each inner automorphism of  $C$  that lifts an automorphism of  $S$  of the kind  $\kappa_Q(x)$  as  $x$  ranges over  $Q$  map  $A$  to itself is redundant.

For technical reasons, we need a stronger notion of Deuring embedding. We will now prepare for the description of this stronger notion.

Thus, let  $C$  be an  $R$ -algebra that contains  $A$  as a subalgebra. Let  $N^{\text{U}(C)}(A)$  denote the *normalizer* of  $A$  in the group  $\text{U}(C)$  of invertible elements of  $C$ ; the group  $N^{\text{U}(C)}(A)$  consists of those  $u \in \text{U}(C)$  such that, for each  $a \in A$ , the member  $uau^{-1}$  of  $C$  already lies in  $A$ . Denote by  $i: \text{U}(A) \rightarrow N^{\text{U}(C)}(A)$  the inclusion.

**Proposition 3.11.** (i) *Conjugation in  $C$  induces a morphism*

$$(1, \eta): (\text{U}(A), N^{\text{U}(C)}(A), i) \longrightarrow (\text{U}(A), \text{Aut}(A), \partial)$$

*of crossed modules, the requisite action of  $N^{\text{U}(C)}(A)$  on  $\text{U}(A)$  being given by conjugation, and hence an  $(N^{\text{U}(C)}(A)/\text{U}(A))$ -normal structure*

$$\eta_{\sharp}: N^{\text{U}(C)}(A)/\text{U}(A) \longrightarrow \text{Out}(A) \quad (3.9)$$

*on  $A$ . Explicitly, the homomorphism  $\eta: N^{\text{U}(C)}(A) \rightarrow \text{Aut}(A)$  is given by the rule*

$$(\eta(u))(a) = uau^{-1} \in A, \quad u \in N^{\text{U}(C)}(A), \quad a \in A.$$

(ii) *The induced homomorphism*

$$\eta_{\flat}: N^{\text{U}(C)}(A)/\text{U}(A) \xrightarrow{\eta_{\sharp}} \text{Out}(A) \xrightarrow{\text{res}} \text{Aut}(S)$$

*maps onto the subgroup  $\kappa_Q(Q) \subseteq \text{Aut}(S)$  if and only if the embedding of  $A$  into  $C$  is a Deuring embedding, that is, if and only if each automorphism  $\kappa_Q(q)$  of  $S$ , as  $q$  ranges over  $Q$ , extends to an inner automorphism of  $C$  that normalizes  $A$ .*

(iii) *If  $A$  coincides with the centralizer of  $S$  in  $C$ , then the induced homomorphism  $\eta_{\flat}$  from  $N^{\text{U}(C)}(A)/\text{U}(A)$  to  $\text{Aut}(S)$  spelled out in (ii) above is injective.*

(iv) *Suppose that the given action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$  lifts to a homomorphism  $\chi: Q \rightarrow N^{\text{U}(C)}(A)/\text{U}(A)$  in the sense that the combined map*

$$Q \xrightarrow{\chi} N^{\text{U}(C)}(A)/\text{U}(A) \xrightarrow{\eta_{\flat}} \text{Aut}(S) \quad (3.10)$$

*coincides with  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ . Then the composite*

$$\sigma: Q \xrightarrow{\chi} N^{\text{U}(C)}(A)/\text{U}(A) \xrightarrow{\eta_{\sharp}} \text{Out}(A) \quad (3.11)$$

of  $\chi$  with the homomorphism (3.9) in (i) above yields a  $Q$ -normal structure  $\sigma$  on  $A$ ; in particular, a strong Deuring embedding structure map  $\chi$  exists and is uniquely determined if  $A$  coincides with the centralizer of  $S$  in  $C$ .

(v) Given a lift  $\chi: Q \rightarrow N^{U(C)}(A)/U(A)$  in the sense that the composite (3.10) thereof with  $\eta_b$  coincides with the structure map  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ , the obvious map from the fiber product group  $\Gamma = N^{U(C)}(A) \times_{N^{U(C)}(A)/U(A)} Q$  to  $Q$  yields a group extension

$$1 \longrightarrow U(A) \xrightarrow{j} \Gamma \longrightarrow Q \longrightarrow 1, \quad (3.12)$$

the injection  $j$  being the obvious homomorphism from  $U(A)$  to  $\Gamma$ , and the obvious action  $\vartheta: \Gamma \rightarrow \text{Aut}(A)$  of  $\Gamma$  on  $A$  (induced by conjugation in  $C$  or, equivalently, by the homomorphism  $\eta: N^{U(C)}(A) \rightarrow \text{Aut}(A)$  in (i) above,) yields a morphism

$$(1, \vartheta): (U(A), \Gamma, j) \longrightarrow (U(A), \text{Aut}(A), \partial) \quad (3.13)$$

of crossed modules, the requisite action of  $\Gamma$  on  $U(A)$  being given by conjugation. The morphism  $(1, \vartheta)$  of crossed modules, in turn, induces the  $Q$ -normal structure (3.11) on  $A$ .

Consider a central  $S$ -algebra  $A$ ; given an algebra  $C$  over  $R = S^Q$ , we define a *strong Deuring embedding* of  $A$  into  $C$  relative to the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$  to be an embedding of  $A$  into  $C$  together with a homomorphism  $\chi: Q \rightarrow N^{U(C)}(A)/U(A)$  such that the combined map

$$Q \xrightarrow{\chi} N^{U(C)}(A)/U(A) \xrightarrow{\eta_b} \text{Aut}(S) \quad (3.14)$$

coincides with the structure map  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ .

Given a strong Deuring embedding  $(A \subseteq C, \chi)$  of the  $S$ -algebra  $A$  into an algebra  $C$  relative to the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on the commutative ring  $S$ , the homomorphism  $\sigma: Q \rightarrow \text{Out}(A)$  given as (3.11) above yields a  $Q$ -normal structure on  $A$ , and we shall say that this  $Q$ -normal structure *arises from the strong Deuring embedding*  $(A \subseteq C, \chi)$  of  $A$  into  $C$  relative to  $\kappa_Q$  or that *the strong Deuring embedding*  $(A \subseteq C, \chi)$  of  $A$  into  $C$  relative to  $\kappa_Q$  *induces the*  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(A)$  on  $A$ .

**Proposition 3.12.** *Let  $C$  be an algebra over  $R = S^Q$ , let  $A \subseteq C$  be an embedding such that  $A$  coincides with the centralizer of  $S$  in  $C$ , and suppose that the embedding is a Deuring embedding with respect to the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$ . Then the data determine a unique group homomorphism  $\chi: Q \rightarrow N^{U(C)}(A)/U(A)$  that turns the embedding  $A \subseteq C$  into a strong Deuring embedding with respect to  $\kappa_Q$ .*

*Proof.* Since  $A$  coincides with the centralizer of  $S$  in  $C$ , the normalizer  $N^{U(C)}(A)$  of  $A$  in  $C$  coincides with the normalizer  $N^{U(C)}(S)$  of  $S$  in  $C$ . The hypothesis that the embedding of  $A$  into  $C$  be a Deuring embedding with respect to the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$ , that is, that each automorphism  $\kappa_Q(q)$  of  $S$ , as  $q$  ranges over  $Q$ , extends to an inner automorphism of  $C$  entails that the canonical homomorphism from the group  $N^{U(C)}(S) = N^{U(C)}(A)$  to  $\text{Aut}(S)$  is a surjective homomorphism onto the subgroup  $\kappa_Q(Q)$  of  $\text{Aut}(S)$ .

Since  $A$  coincides with the centralizer of  $S$  in  $C$ , by Proposition 3.11(iii), the homomorphism  $\eta_b: N^{U(C)}(A)/U(A) \rightarrow \text{Aut}(S)$  is injective. Consequently the structure map  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  lifts to a uniquely determined homomorphism  $\chi: Q \rightarrow N^{U(C)}(A)/U(A)$  in the sense that the composite (3.14) coincides with  $\kappa_Q$ .  $\square$

Recall that the Teichmüller class of a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  is represented by the associated crossed 2-fold extension  $e_{(A, \sigma)}$  introduced as (3.8) above.

**Proposition 3.13.** *If a  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(A)$  on a central  $S$ -algebra  $A$  arises from a strong Deuring embedding  $(C, \chi)$  of  $A$  into an algebra  $C$  over  $R = S^Q$ , then the Teichmüller class  $[e_{(A, \sigma)}] \in H^3(Q, U(S))$  of  $(A, \sigma)$  is zero.*

*Proof.* This is a consequence of Proposition 3.11(v). Indeed,  $(U(S) \times U(A), \Gamma, \partial)$  being endowed with the obvious crossed module structure, let  $e$  denote the associated crossed 2-fold extension

$$e: 0 \longrightarrow U(S) \longrightarrow U(S) \times U(A) \xrightarrow{\partial} \Gamma \longrightarrow Q \longrightarrow 1.$$

There are obvious congruences  $(1, \cdot, \cdot, 1): e \rightarrow e_{(A, \sigma)}$  and  $(1, \cdot, \cdot, 1): e \rightarrow e_0$  of crossed 2-fold extensions whence the assertion.  $\square$

*Remark 3.14.* The morphism (3.13) of crossed modules induces the morphism

$$(\text{Id}, \cdot): (U(A), \Gamma, j) \longrightarrow (U(A), B^\psi, \partial^\psi)$$

of crossed modules. This morphism of crossed modules displays the fact that, in the language of abstract kernels, cf. [Mac67, Section IV.8 p. 124], the abstract kernel associated to the crossed module  $(U(A), B^\psi, \partial^\psi)$  is an extendible kernel. Consequently its obstruction class in  $H^3(Q, U(S))$  vanishes. This obstruction class coincides with the associated Teichmüller class.

It seems worthwhile spelling out a special case of Proposition 3.13.

**Proposition 3.15.** *Let  $C$  be an algebra over  $R = S^Q$  that contains  $A$  as a subalgebra in such a way that  $A$  coincides with the centralizer of  $S$  in  $C$ , and suppose that the embedding of  $A$  into  $C$  is a Deuring embedding with respect to the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$ . Then the  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(A)$  on  $A$  induced by the Deuring embedding of  $A$  into  $C$  via the associated homomorphism  $\chi: Q \rightarrow N^{U(C)}(A)/U(A)$  in Proposition 3.12 above has zero Teichmüller class  $[e_{(A, \sigma)}] \in H^3(Q, U(S))$ .*

### 3.10 The normal algebra associated to a generalized normal Azumaya algebra

Until the end of this section, given an algebra  $A$ , a left  $A$ -module  ${}_A M$  and a right  $A$ -module  $M_A$ , we will use the notation  ${}_A \text{End}({}_A M)$  for the algebra of left  $A$ -endomorphisms and  $\text{End}_A(M_A)$  for the algebra of right  $A$ -endomorphism of  $M_A$ ; accordingly we use the notation  ${}_A \text{Hom}(\cdot, \cdot)$  and  $\text{Hom}_A(\cdot, \cdot)$ .

As before, let  $R = S^Q \subseteq S$ . Consider an Azumaya  $S$ -algebra  $A$ , viewed as an object of  $\mathcal{B}_{S, Q}$ , let  $Q_A \subseteq Q$  denote the image of the canonical homomorphism  $\text{Aut}_{\mathcal{B}_{S, Q}}(A) \longrightarrow Q$ , and consider the associated group extension

$$e_A^{\text{Pic}(S)}: 1 \longrightarrow \text{Pic}(S) \longrightarrow \text{Aut}_{\mathcal{B}_{S, Q}}(A) \longrightarrow Q_A \longrightarrow 1 \quad (3.15)$$

of the kind (2.12), with  $Q_A$  rather than  $Q$ . Then  $A$  represents a member of  $B(S, Q_A)$ , and  $Q_A = Q$  if and only if  $A$  represents a member of  $B(S, Q)$ . Let  $\sigma_A: Q_A \rightarrow \text{Aut}_{\mathcal{B}_{S, Q}}(A)$  be a section for (3.15) of the underlying sets that sends the neutral element of  $Q_A$  to the neutral element of  $\text{Aut}_{\mathcal{B}_{S, Q}}(A)$ , not necessarily a homomorphism.

To simplify the exposition, we will now suppose that  $\kappa_Q$  is injective. The general case then results from the special case with the subgroup  $\kappa_Q(Q)$  of  $\text{Aut}(S)$  substituted for  $Q$ .

For each  $x \in Q_A$ , the value  $\sigma_A(x) \in \text{Aut}_{\mathcal{B}_{S,Q}}(A)$  is an isomorphism class of an invertible  $(A, A)$ -bimodule of grade  $x \in Q$ , and we choose an invertible  $(A, A)$ -bimodule  $M_x$  in  $\sigma_A(x)$ ; while  $M_x$  depends on  $\sigma_A$ , we do not indicate this dependence in notation, to simplify the exposition. In particular,  $M_e$  denotes the algebra  $A$ , viewed as an  $(A, A)$ -bimodule in the canonical way. Let  $M_{\sigma_A} = \bigoplus_{z \in Q_A} M_z$  and  $C_{\sigma_A} = {}_A \text{End}(M_{\sigma_A})$ , the algebra of left  $A$ -endomorphisms of  $M_{\sigma_A}$ . When the group  $Q_A$  is finite,  $C_{\sigma_A}$  (as well as  $C_{\sigma_A}^{\text{op}}$ ) is an Azumaya  $S$ -algebra.

*Remark 3.16.* The construction of  $C_{\sigma_A}$  may be found in the proof of [FW00, Theorem 4.1 (ii)] where it is the basic tool to establish the surjectivity of a homomorphism spelled out below as (10.4) in the case where the group  $Q$  is finite. See also Remark 10.11 below.

Recall that, given a ring automorphism  $\alpha$  of  $C_{\sigma_A}^{\text{op}}$ , the notation  $(M_{\sigma_A})_\alpha$  refers to the right  $C_{\sigma_A}^{\text{op}}$ -module structure on  $M_{\sigma_A}$  given by

$$M_{\sigma_A} \otimes C_{\sigma_A}^{\text{op}} \longrightarrow M_{\sigma_A}, \quad m \cdot b = m\alpha(b), \quad m \in M_{\sigma_A}, \quad b \in C_{\sigma_A}^{\text{op}}.$$

**Proposition 3.17.** *Given a member  $\alpha$  of  $\text{Aut}(C_{\sigma_A}^{\text{op}}, Q)$ , the operation*

$$A \otimes \text{Hom}_{C_{\sigma_A}^{\text{op}}}(M_{\sigma_A}, (M_{\sigma_A})_\alpha) \otimes A \longrightarrow \text{Hom}_{C_{\sigma_A}^{\text{op}}}(M_{\sigma_A}, (M_{\sigma_A})_\alpha)$$

*given by  $a_1 \otimes \varphi \otimes a_2 \longmapsto a_1 \circ \varphi \circ a_2$  ( $a_1, a_2 \in A$ ,  $\varphi \in \text{Hom}_{C_{\sigma_A}^{\text{op}}}(M_{\sigma_A}, (M_{\sigma_A})_\alpha)$ ) where, for  $m \in M_{\sigma_A}$ , the value  $(a_1 \circ \varphi \circ a_2)(m)$  equals  $a_1(\varphi(a_2 m))$ , yields an obvious invertible  $(A, A)$ -bimodule structure on*

$$N_\alpha = \text{Hom}_{C_{\sigma_A}^{\text{op}}}(M_{\sigma_A}, (M_{\sigma_A})_\alpha) \tag{3.16}$$

*of grade equal to the grade of  $\alpha$  in such a way that the assignment to  $\alpha$  of  $N_\alpha$  induces an injective homomorphism  $\Theta_{C_{\sigma_A}} : \text{Out}(C_{\sigma_A}, Q) \rightarrow \text{Aut}_{\mathcal{B}_{S,Q}}(A)$  that is compatible with the grades in  $Q_A$ .*

We postpone the proof after Theorem 3.19 below.

*Remark 3.18.* Suppose that the group  $Q_A$  is finite. Then, by construction, the  $(A, C_{\sigma_A}^{\text{op}})$ -module  $M_{\sigma_A}$  yields an isomorphism  $M_{\sigma_A} : C_{\sigma_A}^{\text{op}} \rightarrow A$  in  $\mathcal{B}_{S,Q}$ . By Proposition 2.1, given a  $(C_{\sigma_A}^{\text{op}}, C_{\sigma_A}^{\text{op}})$ -bimodule  $N_x$  of grade  $x \in Q$ , the  $(A, A)$ -bimodule

$$M_{\sigma_A} \otimes_{C_{\sigma_A}^{\text{op}}} N_x \otimes_{C_{\sigma_A}^{\text{op}}} M_{\sigma_A}^* \cong \text{Hom}_{C_{\sigma_A}^{\text{op}}}(M_{\sigma_A}, M_{\sigma_A} \otimes_{C_{\sigma_A}^{\text{op}}} N_x) \tag{3.17}$$

of grade  $x$  spelled out in Proposition 2.1 represents the image of  $[N_x] \in \text{Aut}_{\mathcal{B}_{S,Q}}(C_{\sigma_A}^{\text{op}})$  in  $\text{Aut}_{\mathcal{B}_{S,Q}}(A)$  under the isomorphism  $(M_{\sigma_A})_* : \text{Aut}_{\mathcal{B}_{S,Q}}(C_{\sigma_A}^{\text{op}}) \longrightarrow \text{Aut}_{\mathcal{B}_{S,Q}}(A)$  induced by the isomorphism  $[M_{\sigma_A}] : C_{\sigma_A}^{\text{op}} \rightarrow A$  in  $\mathcal{B}_{S,Q}$ .

**Theorem 3.19.** *The central  $S$ -algebras  $C_{\sigma_A}$  and  $C_{\sigma_A}^{\text{op}}$  are weakly  $Q_A$ -normal. Furthermore, if the section  $\sigma_A : Q_A \rightarrow \text{Aut}_{\mathcal{B}_{S,Q}}(A)$ , of the underlying sets, for (3.15) is a homomorphism and hence defines a generalized  $Q_A$ -normal structure on  $A$ , this section  $\sigma_A$  determines a  $Q_A$ -normal structure*

$$\sigma_{C_{\sigma_A}} : Q_A \longrightarrow \text{Out}(C_{\sigma_A}, Q)$$

on  $C_{\sigma_A}$  such that

$$\Theta_{C_{\sigma_A}} \circ \sigma_{C_{\sigma_A}} = \sigma_A: Q_A \rightarrow \text{Aut}_{\mathcal{B}_{S,Q}}(A),$$

and the  $Q_A$ -normal  $S$ -algebra  $(C_{\sigma_A}, \sigma_{C_{\sigma_A}})$  is then determined by  $(A, \sigma_A)$  up to within isomorphism.

**Corollary 3.20.** *When the group  $\kappa_Q(Q)$  is finite, the injection  $\text{B}(S, Q) \rightarrow \text{H}^0(Q, \text{B}(S))$  is the identity.*  $\square$

*Remark 3.21.* The main difference between  $A$  and  $C_{\sigma_A}$  is that, while the canonical homomorphism from  $\text{Out}(A, Q_A)$  to  $Q_A$  is not necessarily surjective, the (grade) homomorphism  $\text{Out}(C_{\sigma_A}, Q_A) \rightarrow Q_A$  is surjective, and  $Q_A$  coincides with the image  $Q_{C_{\sigma_A}} \subseteq Q$  of the grade homomorphism  $\text{Out}(C_{\sigma_A}, Q) \rightarrow Q$ .

Given a generalized  $Q$ -normal Azumaya algebra  $(A, \sigma_A: Q \rightarrow \text{Aut}_{\mathcal{B}_{S,Q}}(A))$ , we refer to a  $Q$ -normal algebra of the kind

$$(C_{\sigma_A}^{\text{op}}, \sigma_{C_{\sigma_A}^{\text{op}}} : Q \rightarrow \text{Out}(C_{\sigma_A}^{\text{op}}, Q)) = ({}_A \text{End}(M_{\sigma_A})^{\text{op}}, \sigma_{C_{\sigma_A}^{\text{op}}} ),$$

the notation being that in Theorem 3.19, as the  $Q$ -normal  $S$ -algebra associated to  $(A, \sigma_A)$ , the definite article being justified by the fact that  $(C_{\sigma_A}^{\text{op}}, \sigma_{C_{\sigma_A}^{\text{op}}})$  is uniquely determined by  $(A, \sigma_A)$  up to within isomorphism.

*Proof of the first assertion of Theorem 3.19.* From the group extension (2.12) of  $Q$  by  $\text{Pic}(S)$  we deduce that, given  $x, y \in Q_A$ , there is a projective rank one  $S$ -module  $J_{x,y}$  and an isomorphism

$$f_{x,y}: M_x \otimes_A M_y \longrightarrow J_{x,y} \otimes M_{xy}$$

of  $(A, A)$ -bimodules of grade  $xy$ . Let  $x \in Q_A$ . Then

$$M_x \otimes_A M_{\sigma_A} = \oplus_{y \in Q_A} M_x \otimes_A M_y.$$

Define an isomorphism

$$\beta_x: M_x \otimes_A M_{\sigma_A} \longrightarrow \oplus_{y \in Q_A} J_{x,y} \otimes M_{xy}$$

of  $(A, C_{\sigma_A}^{\text{op}})$ -modules by

$$\beta_x(m_x \otimes m_y) = f_{x,y}(m_x \otimes m_y) \in J_{x,y} \otimes M_{xy}, \quad m_x \in M_x, m_y \in M_y, y \in Q_A.$$

The canonical isomorphisms

$$\text{can}_1: C_{\sigma_A} \longrightarrow {}_A \text{End}(M_x \otimes_A M_{\sigma_A}), \quad \text{can}_2: C_{\sigma_A} \rightarrow {}_A \text{End}(\oplus_{y \in Q_A} J_{x,y} \otimes M_{xy})$$

that are induced by the invertible  $(A, C_{\sigma_A}^{\text{op}})$ -bimodule structures are given by

$$\text{can}_1(f)(m_x \otimes m_y) = m_x \otimes f(m_y), \quad \text{can}_2(f)(j \otimes m_z) = j \otimes f(m_z),$$

where  $f \in C_{\sigma_A} = {}_A \text{End}(M_{\sigma_A})$ ,  $m_x \in M_x$ ,  $m_y \in M_y$ ,  $m_z \in M_z$ ,  $j \in J_{x,y}$ ,  $y, z \in Q$ . The automorphism  $\alpha_x$  of  $C_{\sigma_A}^{\text{op}}$  that makes the diagram

$$\begin{array}{ccc} C_{\sigma_A}^{\text{op}} & \xrightarrow{\text{can}_1} & {}_A \text{End}(M_x \otimes_A M_{\sigma_A}) \\ \alpha_x \downarrow & & \downarrow \beta_{x,*} \\ C_{\sigma_A}^{\text{op}} & \xrightarrow{\text{can}_2} & {}_A \text{End}(\oplus_{y \in Q_A} J_{x,y} \otimes M_{xy}) \end{array}$$

commutative yields an automorphism of  $C_{\sigma_A}^{\text{op}}$  that extends the automorphism  $\kappa_Q(x)$  of  $S$ . Indeed, given  $\varphi \in {}_A \text{End}(M_x \otimes_A M_{\sigma_A})$ , the value  $\beta_{x,*}(\varphi) \in {}_A \text{End}(\oplus_{y \in Q_A} J_{x,y} \otimes M_{xy})$  makes the diagram

$$\begin{array}{ccc} M_x \otimes_A M_{\sigma_A} & \xrightarrow{\varphi} & M_x \otimes_A M_{\sigma_A} \\ \beta_x \downarrow & & \beta_x \downarrow \\ \oplus_{y \in Q_A} J_{x,y} \otimes M_{xy} & \xrightarrow{\beta_{x,*}(\varphi)} & \oplus_{y \in Q_A} J_{x,y} \otimes M_{xy} \end{array}$$

commutative. In the standard manner, the assignment to  $s \in S$  of the  $A$ -linear endomorphism  $f_s: M_{\sigma_A} \rightarrow M_{\sigma_A}$  given by  $f_s(m) = sm$  (via the left  $A$ -module structure on  $M_{\sigma_A}$ ) as  $m$  ranges over  $M_{\sigma_A}$  embeds  $S$  into  $C_{\sigma_A} = {}_A \text{End}(M_{\sigma_A})$ . Since, given  $y \in Q_A$ , the isomorphism  $f_{x,y}$  is one of  $(A, A)$ -bimodules of grade  $xy$ , given  $s \in S$  and  $m_x \in M_x$ ,  $m_y \in M_y$ ,  $j \in J_{x,y}$ ,

$$\begin{aligned} \text{can}_1(f_s)(m_x \otimes m_y) &= m_x \otimes f_s(m_y) = m_x \otimes (sm_y) = m_x s \otimes m_y = ({}^x s) m_x \otimes m_y \\ \beta_x(\text{can}_1(f_s)(m_x \otimes m_y)) &= \beta_x({}^x s m_x \otimes m_y) = f_{x,y}({}^x s m_x \otimes m_y) = ({}^x s) f_{x,y}(m_x \otimes m_y) \\ \text{can}_2(f_{x_s})(j \otimes m_z) &= j \otimes ({}^x s) m_z = ({}^x s)(j \otimes m_z) \end{aligned}$$

we conclude

$$\beta_{x,*}(\text{can}_1(f_s)) = f_{x_s}.$$

Hence the automorphism  $\alpha_x$  of  $C_{\sigma_A}$  extends the automorphism  $\kappa_Q(x)$  of  $S$ . Since  $x \in Q_A$  is arbitrary, the algebras  $C_{\sigma_A}$  and  $C_{\sigma_A}^{\text{op}}$  are weakly  $Q_A$ -normal.  $\square$

We now prepare for the proof of the ‘‘Furthermore’’ statement of Theorem 3.19. We view  $M_{\sigma_A}$  as an  $(A, C_{\sigma_A}^{\text{op}})$ -bimodule in the obvious manner. By assumption, for each  $x \in Q_A$ , the left  $A$ -module structure on  $M_x$  induces an isomorphism  $A \rightarrow \text{End}_{A^{\text{op}}}(M_x)$  and the right  $A$ -module structure on  $M_x$  induces an isomorphism  $A^{\text{op}} \rightarrow {}_A \text{End}(M_x)$ . These right  $A$ -module structures induce an injection  $A^{\text{op}} \rightarrow C_{\sigma_A}$  of  $S$ -algebras.

Given  $x, y \in Q_A$ , the  $(A, A)$ -bimodule  ${}_A \text{Hom}(M_x, M_y) \cong M_x^* \otimes_A M_y$ , necessarily of grade  $x^{-1}y$ , is an  $S$ -submodule of  $C_{\sigma_A}$  in an obvious manner. Let  $x, y, z \in Q_A$ ; given  $h_{y,x}: M_x \rightarrow M_y$  and  $h_{z,y}: M_y \rightarrow M_z$ , the composite  $h_{z,y} \circ h_{y,x}: M_x \rightarrow M_z$  is defined so that, as  $S$ -modules,

$$C_{\sigma_A} = \prod_{u \in Q_A} \bigoplus_{v \in Q_A} {}_A \text{Hom}(M_u, M_v) \cong \prod_{u \in Q_A} {}_A \text{Hom}(M_u, M_{\sigma_A}). \quad (3.18)$$

Thus, with a grain of salt, we can think of the members of  $C_{\sigma_A}$  as being matrices whose columns have only finitely many non-zero entries.

**Lemma 3.22.** *The injection  $A \rightarrow C_{\sigma_A} \text{End}(M_{\sigma_A}) = \text{End}_{C_{\sigma_A}^{\text{op}}}(M_{\sigma_A})$  given by the assignment to  $a \in A$  of  $f_a \in C_{\sigma_A} \text{End}(M_{\sigma_A})$  where  $f_a(m) = am$  ( $m \in M_{\sigma_A}$ ) is an isomorphism.*

In the case where the group  $Q_A$  is finite the claim of the lemma is immediate, but for general  $Q_A$  we must be more circumspect.

*Proof.* Let  $f: M_{\sigma_A} \rightarrow M_{\sigma_A}$  be  $C_{\sigma_A}$ -linear and, given  $x, w, z \in Q_A$ , consider the restrictions  $f: M_x \rightarrow M_z$  and  $f: M_y \rightarrow M_w$ . Given  $y \in Q$  and  $h_{x,y} \in {}_A \text{Hom}(M_x, M_y)$ , since  $f$  is  $C_{\sigma_A}$ -linear, the diagram

$$\begin{array}{ccc} M_x & \xrightarrow{f} & M_z \\ h_{x,y} \downarrow & & h_{x,y} \downarrow \\ M_y & \xrightarrow{f} & M_w \end{array}$$

is commutative.

Composition of endomorphisms yields a canonical isomorphism

$${}_A \text{Hom}(M_x, M_y) \otimes_A ({}_A \text{Hom}(M_y, M_x)) \longrightarrow {}_A \text{End}(M_x) \cong A^{\text{op}}$$

of  $(A, A)$ -bimodules whence there are finitely many  $h_{x,y}^j \in {}_A \text{Hom}(M_x, M_y)$  and  $\tilde{h}_{y,x}^j \in {}_A \text{Hom}(M_y, M_x)$  such that  $\sum \tilde{h}_{y,x}^j \circ h_{x,y}^j = \text{Id}_{M_x}$ . Accordingly, the diagram

$$\begin{array}{ccc} M_x & \xrightarrow{f} & M_z \\ \sum \tilde{h}_{y,x}^j \circ h_{x,y}^j \downarrow & & \downarrow \sum \tilde{h}_{y,x}^j \circ h_{x,y}^j \\ M_x & \xrightarrow{f} & M_z \end{array}$$

is commutative. However the right-hand vertical arrow is zero unless  $x = z$ . Thus only the components of  $f$  of the kind  $M_x \rightarrow M_x$  are non-zero and, since  $f$  is  $C_{\sigma_A}$ -linear, relative to the embedding of  $A^{\text{op}}$  into  $C_{\sigma_A}$ , the endomorphism  $f: M_x \rightarrow M_x$  is  $A^{\text{op}}$ -linear and hence, in view of the  $S$ -algebra isomorphism  $A \rightarrow \text{End}_{A^{\text{op}}}(M_x)$ , on  $M_x$ , the endomorphism  $f$  is given by left multiplication by a uniquely determined  $a_x \in A$  so that  $f(m) = a_x m$ .

Now, given  $x, y \in Q_A$ , and  $h_{x,y} \in {}_A \text{Hom}(M_x, M_y)$ , the diagram

$$\begin{array}{ccc} M_x & \xrightarrow{f_{a_x}} & M_x \\ h_{x,y} \downarrow & & h_{x,y} \downarrow \\ M_y & \xrightarrow{f_{a_y}} & M_y \end{array} \quad (3.19)$$

is commutative.

However, the evaluation map

$${}_A \text{Hom}(M_x, M_y) \otimes_A M_x \longrightarrow M_y$$

is an isomorphism. Hence, given  $m \in M_y$ , there are  $b_1, \dots, b_k \in {}_A \text{Hom}(M_x, M_y)$  and  $m_1, \dots, m_k \in M_x$  such that  $m = \sum b_j m_j$ . Since the diagram (3.19) is commutative,

$$f_{a_y}(m) = \sum f_{a_y} b_j m_j = \sum b_j f_{a_x} m_j.$$

On the other hand,

$$\sum b_j f_{a_x} m_j = \sum f_{a_x} b_j m_j = f_{a_x} \sum b_j m_j = f_{a_x}(m).$$

Since  $m \in M_y$  is arbitrary, we conclude  $f_{a_x} = f_{a_y}$ , whence the injection  $A \rightarrow \text{End}_{C_{\sigma_A}}(M_{\sigma_A})$  is surjective as asserted.  $\square$

**Corollary 3.23.** *Given an invertible  $(A, A)$ -bimodule  $N$ , the injection*

$$\iota_N: N \longrightarrow \text{Hom}_{C_{\sigma_A}^{\text{op}}}(M_{\sigma_A}, N \otimes_A M_{\sigma_A}), \quad f_m(w) = m \otimes w,$$

*is an isomorphism of  $(A, A)$ -bimodules.*

*Proof.* The injection  $\iota_N$  is the image of  $\text{Id}_{N \otimes_A M_{\sigma_A}}$  under the adjointness isomorphism

$$\text{ad}: \text{End}_{C_{\sigma_A}^{\text{op}}} (N \otimes_A M_{\sigma_A}) \longrightarrow \text{Hom}_A(N, \text{Hom}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}, N \otimes_A M_{\sigma_A})). \quad (3.20)$$

Since, as a right  $A$ -module,  $N$  is finitely generated projective, evaluation yields an isomorphism

$$\text{ev}: N \otimes_A (\text{Hom}_A(N, \text{Hom}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}, N \otimes_A M_{\sigma_A}))) \longrightarrow \text{Hom}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}, N \otimes_A M_{\sigma_A}) \quad (3.21)$$

of  $(A, A)$ -bimodules. In view of Lemma 3.22, the canonical isomorphism

$$\text{End}_{C_{\sigma_A}^{\text{op}}} (N \otimes_A M_{\sigma_A}) \cong \text{End}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}),$$

combined with the isomorphism  $A \rightarrow \text{End}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A})$ , identifies  $\text{End}_{C_{\sigma_A}^{\text{op}}} (N \otimes_A M_{\sigma_A})$  with  $A$ . Up to this identification, the injection  $\iota_N$  is the composite

$$N \otimes_A \text{End}_{C_{\sigma_A}^{\text{op}}} (N \otimes_A M_{\sigma_A}) \xrightarrow{\text{ev} \circ (\text{Id}_N \otimes_A \text{ad})} \text{Hom}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}, N \otimes_A M_{\sigma_A}).$$

□

*Proof of Proposition 3.17.* Given  $\varphi \in N_\alpha = \text{Hom}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}, (M_{\sigma_A})_\alpha)$ ,  $m \in M_{\sigma_A}$ , and  $s \in S$ ,

$$(\varphi \circ s)(m) = \varphi(s(m));$$

however,  $f_s(m) = sm$  ( $m \in M_{\sigma_A}$ ) defines a member  $f_s$  of  $C_{\sigma_A}^{\text{op}} = {}_A \text{End}(M_{\sigma_A})^{\text{op}}$  and, by the definition of  $(M_{\sigma_A})_\alpha$ ,

$$\varphi(sm) = \varphi(mb_s) = \varphi(f_s(m)) = f_{\alpha(s)}\varphi(m) = \varphi(m)b_{\alpha(s)} = \alpha(s)\varphi(m)$$

whence  $\varphi \circ s = \alpha(s) \circ \varphi$ . Furthermore,

$$\begin{aligned} \text{Hom}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}, (M_{\sigma_A})_\alpha) \otimes_A \text{Hom}_{C_{\sigma_A}^{\text{op}}} ((M_{\sigma_A})_\alpha, M_{\sigma_A}) &\cong \text{End}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}) \cong A, \\ \text{Hom}_{C_{\sigma_A}^{\text{op}}} ((M_{\sigma_A})_\alpha, M_{\sigma_A}) \otimes_A \text{Hom}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}, (M_{\sigma_A})_\alpha) &\cong \text{End}_{C_{\sigma_A}^{\text{op}}} ((M_{\sigma_A})_\alpha) \cong A. \end{aligned}$$

Consequently  $N_\alpha$  is an invertible  $(A, A)$ -bimodule of grade equal to the grade of  $\alpha$  as asserted. We leave the proof of the remaining claims to the reader. □

*Proof of the “Furthermore” statement of Theorem 3.19.* Suppose that the given section  $\sigma_A$  from  $Q$  to  $\text{Aut}_{\mathcal{B}_{S,Q}}(A)$  is a homomorphism of groups. Fix  $x \in Q_A$ . The construction of the extension  $\alpha_x: C_{\sigma_A} \rightarrow C_{\sigma_A}$  of the automorphism  $\kappa_Q(x)$  of  $S$  in the proof of the first statement simplifies since now the constituents  $J_{x,y}$  are trivial, i. e., copies of the commutative ring  $S$ . Thus, given  $y \in Q_A$ , there is an isomorphism

$$f_{x,y}: M_x \otimes_A M_y \longrightarrow M_{xy}$$

of (invertible)  $(A, A)$ -bimodules. Then

$$M_x \otimes_A M_{\sigma_A} = \bigoplus_{y \in Q_A} M_x \otimes_A M_y.$$



Define an isomorphism

$$\beta_x: M_x \otimes_A M_{\sigma_A} \longrightarrow \oplus_{y \in Q_A} M_{xy} = M_{\sigma_A},$$

of  $(A, C_{\sigma_A}^{\text{op}})$ -modules by

$$\beta_x(m_x \otimes m_y) = f_{x,y}(m_x \otimes m_y) \in M_{xy}, \quad m_x \in M_x, \quad m_y \in M_y, \quad y \in Q_A.$$

The canonical isomorphism

$$\text{can}_x: C_{\sigma_A} \longrightarrow {}_A \text{End}(M_x \otimes_A M_{\sigma_A})$$

that is induced by the invertible  $(A, C_{\sigma_A}^{\text{op}})$ -bimodule structure on  $M_x \otimes_A M_{\sigma_A}$  is given by

$$\text{can}_x(f)(m_x \otimes m_y) = m_x \otimes f(m_y), \quad m_x \in M_x, \quad m_y \in M_y, \quad f \in C_{\sigma_A} = {}_A \text{End}(M_{\sigma_A}).$$

The automorphism  $\alpha_x$  of  $C_{\sigma_A}$  that makes the diagram

$$\begin{array}{ccc} C_{\sigma_A} & \xrightarrow{\text{can}_x} & {}_A \text{End}(M_x \otimes_A M_{\sigma_A}) \\ \alpha_x \downarrow & & \downarrow \beta_{x,*} \\ C_{\sigma_A} & \xlongequal{\quad} & {}_A \text{End}(M_{\sigma_A}) \end{array}$$

commutative yields an automorphism of  $C_{\sigma_A}$  that extends the automorphism  $\kappa_Q(x)$  of  $S$ . Now, given  $\varphi \in {}_A \text{End}(M_x \otimes_A M_{\sigma_A})$ , the value  $\beta_{x,*}(\varphi)$  makes the diagram

$$\begin{array}{ccc} M_x \otimes_A M_{\sigma_A} & \xrightarrow{\varphi} & M_x \otimes_A M_{\sigma_A} \\ \beta_x \downarrow & & \downarrow \beta_x \\ M_{\sigma_A} & \xrightarrow{\beta_{x,*}(\varphi)} & M_{\sigma_A} \end{array}$$

commutative. Hence, given  $b \in C_{\sigma_A} = {}_A \text{End}(M_{\sigma_A})$ ,

$$(\alpha_x(b))(\beta_x(m_x \otimes m)) = \beta_x(m_x \otimes bm). \quad (3.22)$$

Consequently, the  $C_{\sigma_A}$ -module structure on  $M_x \otimes_A M_{\sigma_A}$  being given by

$$C_{\sigma_A} \otimes (M_x \otimes_A M_{\sigma_A}) \longrightarrow M_x \otimes_A M_{\sigma_A}, \quad f \otimes m_x \otimes m \longmapsto m_x \otimes f(m),$$

the isomorphism  $\beta_x$  is one of left  $C_{\sigma_A}$ -modules from  $M_x \otimes_A M_{\sigma_A}$  onto  ${}_{\alpha_x} M_{\sigma_A}$  or, equivalently, one of right  $C_{\sigma_A}^{\text{op}}$ -modules from  $M_x \otimes_A M_{\sigma_A}$  onto  $(M_{\sigma_A})_{\alpha_x}$ . We define the value  $\sigma_{C_{\sigma_A}}(x) \in \text{Out}(C_{\sigma_A}, Q)$  to be the class of  $\alpha_x$  in  $\text{Out}(C_{\sigma_A}, Q)$ .

By Proposition 3.17, the homomorphism  $\Theta_{C_{\sigma_A}}: \text{Out}(C_{\sigma_A}, Q) \rightarrow \text{Aut}_{\mathcal{B}_{S,Q}}(A)$  sends the class of  $\alpha_x$  to the isomorphism class of the invertible  $(A, A)$ -bimodule

$$N_{\alpha_x} = \text{Hom}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}, (M_{\sigma_A})_{\alpha_x}) \cong \text{Hom}_{C_{\sigma_A}^{\text{op}}} (M_{\sigma_A}, M_x \otimes_A M_{\sigma_A}) \cong M_x,$$

the last isomorphism being a consequence of Corollary 3.23. Hence  $\Theta_{C_{\sigma_A}}$  sends the class of  $\alpha_x$  to the isomorphism class of the  $(A, A)$ -bimodule  $M_x$ . Consequently  $\Theta_{C_{\sigma_A}} \circ \sigma_{C_{\sigma_A}} = \sigma_A$ . Furthermore, since  $\Theta_{C_{\sigma_A}}$  is injective and since  $\sigma_A$  is a homomorphism, the section  $\sigma_{C_{\sigma_A}}$ , at first one of the underlying sets, is a homomorphism.

We leave the proof of the claim that the  $Q_A$ -normal  $S$ -algebra  $(C_{\sigma_A}, \sigma_{C_{\sigma_A}})$  is determined by  $(A, \sigma_A)$  up to within isomorphism to the reader.  $\square$

### 3.11 The Teichmüller class of a generalized $Q$ -normal Azumaya algebra

Let  $(A, \sigma_A: Q \rightarrow \text{Aut}_{\mathcal{B}_{S,Q}}(A))$  be a generalized  $Q$ -normal Azumaya algebra, and let

$$(C^{\text{op}}, \sigma_{C^{\text{op}}}) = (C_{\sigma_A}^{\text{op}}, \sigma_{C_{\sigma_A}^{\text{op}}} : Q \rightarrow \text{Out}(C_{\sigma_A}^{\text{op}}, Q)) = ({}_A \text{End}(M_{\sigma_A})^{\text{op}}, \sigma_{C_{\sigma_A}^{\text{op}}})$$

be the  $Q$ -normal  $S$ -algebra associated to  $(A, \sigma_A)$ . We then refer to the Teichmüller complex  $e_{(C^{\text{op}}, \sigma_{C^{\text{op}}})}$  of the  $Q$ -normal  $S$ -algebra  $(C^{\text{op}}, \sigma_{C^{\text{op}}})$ , cf. (3.8), as the *Teichmüller complex* of  $(A, \sigma_A)$  and to the class  $[e_{(C^{\text{op}}, \sigma_{C^{\text{op}}})}] \in H^3(Q, U(S))$  as the *Teichmüller class* of  $(A, \sigma_A)$ .

**Theorem 3.24.** *The assignment to a generalized  $Q$ -normal Azumaya algebra  $(A, \sigma_A)$  of its Teichmüller class  $[e_{(C^{\text{op}}, \sigma_{C^{\text{op}}})}] \in H^3(Q, U(S))$  yields a homomorphism*

$$t: k\mathcal{R}ep(Q, \mathcal{B}_{S,Q}) \longrightarrow H^3(Q, U(S)) \quad (3.23)$$

of abelian groups such that, when the generalized  $Q$ -normal Azumaya  $S$ -algebra  $(A, \sigma_A)$  arises from an ordinary  $Q$ -normal algebra structure  $\sigma: Q \rightarrow \text{Out}(A, Q)$  on  $A$ ,

$$[e_{(C^{\text{op}}, \sigma_{C^{\text{op}}})}] = [e_{(A, \sigma)}] \in H^3(Q, U(S)). \quad (3.24)$$

*Remark 3.25.* A variant of the homomorphism  $t$ , written there as  $\chi$ , is given in [FW00, Theorem 3.4 (ii)] by the assignment to a generalized  $Q$ -normal Azumaya  $S$ -algebra of an explicit 3-cocycle of  $Q$  with values in  $U(S)$ . The identity (3.24) is equivalent to the statement of [FW00, Theorem 3.4 (iii)].

*Proof of the first assertion Theorem 3.24.* Proposition 3.9, suitably adjusted to the present situation, entails that  $t$  is a homomorphism. We leave the details to the reader.  $\square$

We postpone the proof of the second assertion of Theorem 3.24 until Subsection 9.2 below.

## 4 Crossed products with normal algebras

As before,  $Q$  denotes a group,  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of  $Q$  on the commutative ring  $S$ , and  $R$  the subring  $R = S^Q \subset S$  of  $S$  that consists of the elements fixed under  $Q$ . In this section, we shall explore certain crossed products of  $Q$  with  $Q$ -normal  $S$ -algebras. Formally, the construction goes back to Teichmüller [Tei40], who carried it out for certain algebras over fields (“Normalringe”).

Consider a central  $S$ -algebra  $A$ . Suppose that there is a group extension

$$e_Q: 1 \longrightarrow K \xrightarrow{j} \Gamma \xrightarrow{\pi} Q \longrightarrow 1 \quad (4.1)$$

together with a morphism

$$(i, \vartheta): (K, \Gamma, j) \longrightarrow (U(A), \text{Aut}(A), \partial) \quad (4.2)$$

of crossed modules having  $i$  injective. Such a morphism of crossed modules, in turn, induces a  $Q$ -normal structure  $\sigma_\vartheta: Q \rightarrow \text{Aut}(A)$  on  $A$ . Conversely, given a  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(A)$  on  $A$ , an extension of the kind (4.1) with  $K = U(A)$  together with a morphism of crossed modules of the kind (4.2) inducing  $\sigma$  exists if and only if the Teichmüller class  $[e_{(A, \sigma)}] \in H^3(Q, U(S))$  of  $(A, \sigma)$  is zero, cf. the proof of Proposition 3.13 as well as

Remark 3.14 above. Thus we consider a  $Q$ -normal algebra  $(A, \sigma)$  having zero Teichmüller class and fix an extension of the kind (4.1) together with a morphism of crossed modules of the kind (4.2).

We remind the reader that the second cohomology group  $H^2(Q, Z_K)$  of  $Q$  with values in the center  $Z_K$  of  $K$  acts faithfully and transitively on the congruence classes of extensions of the kind (4.1) that have the same outer action  $Q \rightarrow \text{Out}(K)$ , cf. [Mac67, Section IV.8 Theorem 8.8 p. 128].

#### 4.1 First crossed product algebra construction

Relative to the action of  $\Gamma$  on  $A$  given by the homomorphism  $\vartheta: \Gamma \rightarrow \text{Aut}(A)$ , let  $A^t\Gamma$  be the twisted group ring of  $\Gamma$  with twisted coefficients in  $A$ , and let  $\langle y - j(y), y \in K \rangle$  denote the two-sided ideal in  $A^t\Gamma$  generated by the elements  $y - j(y) \in A^tK (\subseteq A^t\Gamma)$  as  $y$  ranges over  $K$ . Define the algebra  $(A, Q, e_Q, \vartheta)$  to be the quotient algebra

$$(A, Q, e_Q, \vartheta) = A^t\Gamma / \langle y - j(y), y \in K \rangle.$$

It is obvious that the ring  $R (= S^Q)$  lies in the center of  $(A, Q, e_Q, \vartheta)$  whence  $(A, Q, e_Q, \vartheta)$  is an  $R$ -algebra. We shall refer to the algebra  $(A, Q, e_Q, \vartheta)$  as the *crossed product* of  $A$  and  $Q$ , with respect to  $e_Q$  and  $\vartheta$ .

#### 4.2 Second crossed product algebra construction

Relative to the group extension (4.1), let  $v: Q \rightarrow \Gamma$  be a section for  $\pi$  of the underlying sets, i.e., a section which is not necessarily a homomorphism. Assume for convenience that  $v(1) = 1$  and, for  $q \in Q$ , write  $v_q = v(q)$ . Let  $\varphi: Q \times Q \rightarrow K$  be a corresponding normalized 2-cocycle relative to  $v$  so that

$$v_p v_q = \varphi(p, q) v_{pq}, \quad p, q \in Q.$$

Then the *crossed product*  $(A, Q, e_Q, \vartheta)$  of  $A$  and  $Q$ , with respect to the 2-cocycle  $\varphi$  and the homomorphism  $\vartheta: \Gamma \rightarrow \text{Aut}(A)$ , is the algebra having  $\oplus_Q A v_q$  as its underlying left  $A$ -module and whose multiplicative structure is given by

$$v_q a = (\vartheta(v_q) a) v_q, \quad v_p v_q = \varphi(p, q) v_{pq}, \quad a \in A, \quad p, q \in Q.$$

In the special case where the group  $Q$  is finite, where  $A = S$ , and where  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ , that notion of crossed product comes down to the ordinary crossed product algebra. The more general construction of  $(A, Q, \varphi, \vartheta)$  was given, in the classical situation (i.e.,  $S$  a field etc.), in [Tei40, p. 145] as well as in [EM48, p. 9], and for Azumaya algebras in [Chi72, p. 13] where  $S|R$  was still assumed to be a Galois extension of commutative rings with Galois group  $Q$ .

#### 4.3 Equivalence of the two crossed product algebra constructions

The two notions of crossed product algebra given above are equivalent:

**Proposition 4.1.** *The association*

$$(A, Q, e_Q, \vartheta) \longrightarrow (A, Q, \varphi, \vartheta), \quad x \longmapsto (x v_{\pi(x)}^{-1}) v_{\pi(x)}, \quad x \in \Gamma, \quad (4.3)$$

where  $xv_{\pi(x)}^{-1} \in K$  is to be interpreted as a member of  $A$ , yields a morphism of algebras and, likewise, the association  $v_q \mapsto v_q$  ( $q \in Q$ ) yields a morphism  $(A, Q, \varphi, \vartheta) \rightarrow (A, Q, e_Q, \vartheta)$  of algebras; these morphisms preserve the structures and are inverse to each other. Hence the section  $v: Q \rightarrow \Gamma$ , in the category of sets, for the surjection  $\pi$  in the extension (4.1) yields a basis of the left  $A$ -module that underlies the crossed product algebra  $(A, Q, e_Q, \vartheta)$ .

*Proof.* The morphism (4.3) of algebras is plainly well defined and surjective.

As an  $A^tK$ -module, the algebra  $A^t\Gamma$  is free, having as basis the family  $\{v_q; q \in Q\} \subseteq \Gamma$ . Furthermore, write the injection  $K \rightarrow A$  as  $u \mapsto a_u$  ( $u \in K$ ); the kernel of the canonical surjection  $A^tK \rightarrow A$  of algebras given by the association  $A^tK \ni au \mapsto aa_u \in A$  is the two-sided ideal in  $A^tK$  generated by the elements  $y - j(y) \in A^tK$  as  $y$  ranges over  $K$ . Now, the  $A$ -module that underlies the algebra  $(A, Q, e_Q, \vartheta)$  arises from  $A^t\Gamma$ , viewed as an  $A^tK$ -module, through the surjection  $A^tK \rightarrow A$  of algebras. By construction, as an  $A$ -module, the algebra  $(A, Q, \varphi, \vartheta)$  has likewise the family  $\{v_q; q \in Q\} \subseteq \Gamma$  as basis whence (4.3) is an isomorphism of algebras.  $\square$

*Remark 4.2.* The classical fact that, up to isomorphism, the crossed product algebra depends only on the congruence class of the corresponding group extension extends to our more general situation in an obvious way; we leave the details to the reader.

#### 4.4 Properties of the crossed product algebra

We will write  $(A, Q, e_Q, \vartheta)$  or  $(A, Q, \varphi, \vartheta)$  according as which construction of the crossed product algebra is more convenient for the particular situation under discussion. In the special case where  $A = S$ , the action  $\vartheta: \Gamma \rightarrow \text{Aut}(S)$  of  $\Gamma$  on  $S$  necessarily coincides with the composite  $\kappa_Q \circ \pi$ . Proofs of the statements below are straightforward and mostly left to the reader.

**Proposition 4.3.** (i) *The algebra  $A = Av_1$  is a subalgebra of the crossed product algebra  $(A, Q, \varphi, \vartheta)$  and lies in the centralizer of  $S$ .*

(ii) *If  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ , the algebra  $A$  coincides with the centralizer of  $S$ . In particular, when  $A = S$ , this comes down to the familiar fact that  $S$  is a maximal commutative subring of the algebra  $(S, Q, \varphi, \kappa_Q \circ \pi)$ .*

(iii) *If  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ , the ring  $R$  coincides with the center of  $(A, Q, \varphi, \vartheta)$ .*

(iv) *The group  $\Gamma$  embeds canonically into the normalizer  $N^{\text{U}(A, Q, \varphi, \vartheta)}(A)$  of  $A$  in  $\text{U}(A, Q, \varphi, \vartheta)$  so that, given  $a = av_1 \in A$ ,  $x \in \Gamma$ ,*

$$xax^{-1} = \vartheta(x)a.$$

*In particular, each automorphism  $\kappa_Q(q)$  of  $S = Sv_1$ , as  $q$  ranges over  $Q$ , extends to an inner automorphism of  $(A, Q, \varphi, \vartheta)$  that normalizes  $A$ . If  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$  and if the injection  $i: K \rightarrow \text{U}(A)$  is an isomorphism of groups, then  $\Gamma = N^{\text{U}(A, Q, \varphi, \vartheta)}(A)$ .*

*Proof of the second statement of (4.3) (ii).* Let  $x = \sum a_q v_q \in (A, Q, \varphi, \vartheta)$  so that, for each  $s \in S$ ,  $xs = sx$ . This implies that, given  $s \in S$  and  $q \in Q$ ,  $a_q v_q s = s v_q a_q$ , and therefore  $(^q s - s)a_q = 0$ . Let  $q \neq 1$ . If  $a_q \neq 0$  then  $Sa_q \subset A$  is a cyclic  $S$ -submodule; let  $J$  denote the annihilator ideal of  $Sa_q$ . Since  $S|R$  is assumed to be a Galois extension of commutative rings with Galois group  $Q$ , by (1.2) (iii), there is an  $s \in S$  with  $^q s - s \notin J$ , and so  $(^q s - s)a_q \neq 0$ .  $\square$

To simplify the notation, we shall denote by  $M_{e_Q}$  the left  $A$ -module that underlies the crossed product algebra  $(A, Q, e_Q, \vartheta)$ ; by construction, as an  $S$ -module,  $M_{e_Q} \cong \oplus_Q Av_q$ . The crossed product algebra structure on  $M_{e_Q}$  yields some additional structure on the central  $S$ -algebra  ${}_A \text{End}(M_{e_Q}) \cong M_{|Q|}(A^{\text{op}})$ :

**Proposition 4.4.** (i) *With respect to the action of the group  $\Gamma$  on  $S$  via the combined map  $\kappa_Q \circ \pi: \Gamma \rightarrow Q \rightarrow \text{Aut}(S)$ , the association*

$$\Gamma \times M_{e_Q} \longrightarrow M_{e_Q}, \quad (x, b) \longmapsto xb \in (A, Q, \varphi, \vartheta), \quad x \in \Gamma, \quad b \in M_{e_Q}, \quad (4.4)$$

*yields an  $S^t\Gamma$ -module structure on  $M_{e_Q}$ .*

(ii) *The induced action  $\beta_1: \Gamma \rightarrow \text{Aut}({}_A \text{End}(M_{e_Q}))$  on  ${}_A \text{End}(M_{e_Q})$  given by*

$$(\beta_1(x)f)b = xf(x^{-1}b), \quad x \in \Gamma, \quad b \in M_{e_Q}, \quad f \in {}_A \text{End}(M_{e_Q}),$$

*is trivial on  $K \subset \Gamma$  and hence induces, on  ${}_A \text{End}(M_{e_Q})$ , a  $Q$ -equivariant structure*

$$\tau_{e_Q}: Q \longrightarrow \text{Aut}({}_A \text{End}(M_{e_Q})). \quad (4.5)$$

(iii) *The rule*

$$x(av_q) = (\vartheta^{(x)}a)v_q, \quad a \in A, \quad x \in \Gamma, \quad q \in Q,$$

*yields another  $S^t\Gamma$ -module structure on  $M_{e_Q}$ .*

(iv) *The action  $\beta_2: \Gamma \rightarrow \text{Aut}({}_A \text{End}(M_{e_Q}))$  on  ${}_A \text{End}(M_{e_Q})$  induced by the  $S^t\Gamma$ -module structure in (iii) is given by the identity*

$$(\beta_2(x)f)(av_q) = a^x(f(v_q)), \quad x \in \Gamma, \quad a \in A, \quad f \in {}_A \text{End}(M_{e_Q}), \quad q \in Q,$$

*and induces a  $Q$ -normal structure  $\sigma_{e_Q, \vartheta}: Q \rightarrow \text{Out}({}_A \text{End}(M_{e_Q}))$  on  ${}_A \text{End}(M_{e_Q})$  which, under the isomorphism  ${}_A \text{End}(M_{e_Q}) \cong M_{|Q|}(A^{\text{op}})$  (cf. Subsections 3.5 and 3.6 above), corresponds to the  $Q$ -normal structure  $\sigma_{\vartheta, |Q|}^{\text{op}}$  on  $M_{|Q|}(A^{\text{op}})$ , i. e., to the  $Q$ -normal structure on  $M_{|Q|}(A^{\text{op}})$  induced by the  $Q$ -normal structure  $\sigma_{\vartheta}: Q \rightarrow \text{Aut}(A)$  on  $A$ .*

(v) *Given  $x \in \Gamma$ , the association  $v_q \longmapsto xv_q \in (A, Q, \varphi, \vartheta)$ , as  $q$  ranges over  $Q$ , yields an  $A$ -linear automorphism  $i_x: M_{e_Q} \rightarrow M_{e_Q}$ , i.e., a member of*

$$\text{U}({}_A \text{End}(M_{e_Q})) = {}_A \text{Aut}(M_{e_Q}) \cong \text{GL}_{|Q|}(A^{\text{op}}).$$

(vi) *The two  $S^t\Gamma$ -structures on  $M_{e_Q}$  given in (i) and (iii) are related by the identity*

$$xb = {}^x(i_x(b)), \quad x \in \Gamma, \quad b \in M_{e_Q}.$$

(vii) *Given  $x \in \Gamma$  and  $f \in \text{End}_A(M_{e_Q})$ ,*

$$\beta_2(x^{-1})(\beta_1(x)f) = i_x f i_x^{-1}.$$

*Thus, in view of (ii), (iv), and (v), the  $Q$ -normal structure  $\sigma_{e_Q, \vartheta}$  on  ${}_A \text{End}(M_{e_Q})$  factors through the  $Q$ -equivariant structure (4.5) on  ${}_A \text{End}(M_{e_Q})$  and is therefore  $Q$ -equivariant.*

(viii) *Given  $u \in (A, Q, \varphi, \vartheta)$ , define  $f_u \in \text{End}_A(M_{e_Q})$  by  $f_u(b) = bu \in (A, Q, \varphi, \vartheta)$ , for  $b \in M_{e_Q}$ . The assignment to  $u^{\text{op}} \in (A, Q, \varphi, \vartheta)^{\text{op}}$  of  $f_u$  yields an injection*

$$(A, Q, \varphi, \vartheta)^{\text{op}} \longrightarrow \text{End}_A(M_{e_Q})$$

which identifies the algebra  $(A, Q, \varphi, \vartheta)^{\text{op}}$  with the subalgebra  ${}_A \text{End}(M_{e_Q})^Q$  of  ${}_A \text{End}(M_{e_Q})$ , i. e., with the subalgebra that consists of the elements fixed under the  $Q$ -equivariant structure (4.5).

(ix) If  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ , then the obvious map

$$\alpha: S \otimes_R (A, Q, \varphi, \vartheta)^{\text{op}} \longrightarrow {}_A \text{End}(M_{e_Q})$$

given by  $[\alpha(s \otimes u^{\text{op}})]b = sbu$ , for  $s \in S$ ,  $b \in M_{e_Q}$ ,  $u \in (A, Q, \varphi, \vartheta)$ , is an isomorphism of  $S$ -algebras as well as, relative to the  $Q$ -equivariant structure (4.5), an isomorphism of  $S^t Q$ -modules. In particular, when  $A = S$ , this statement recovers the familiar fact that the crossed product  $R$ -algebra  $(S, Q, \varphi, \kappa_Q \circ \pi)$  is split by  $S$ .

(x) If  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ , then the induced isomorphism

$$\alpha_{\sharp}: \text{Aut}(S \otimes_R (A, Q, \varphi, \vartheta)^{\text{op}}) \longrightarrow \text{Aut}({}_A \text{End}(M_{e_Q}))$$

identifies the obvious  $Q$ -equivariant structure on  $S \otimes_R (A, Q, \varphi, \vartheta)^{\text{op}}$  which comes from scalar extension with the  $Q$ -equivariant structure (4.5) on  ${}_A \text{End}(M_{e_Q})$ . Consequently the  $Q$ -normal algebra  $({}_A \text{End}(M_{e_Q}), \sigma_{e_Q, \vartheta})$  then arises from the  $R$ -algebra  $(A, Q, \varphi, \vartheta)^{\text{op}}$  by scalar extension.

(xi) If  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$  and if  $A$  is an Azumaya  $S$ -algebra, then  $(A, Q, \varphi, \vartheta)$  is an Azumaya  $R$ -algebra.

Concerning statement (x), we note that statement (vii) already implies that the  $Q$ -normal algebra  $({}_A \text{End}(M_{e_Q}), \sigma_{e_Q, \vartheta})$  arises by scalar extension but statement (x) is more precise.

*Proof.* (vii): Let  $a \in A$ ,  $x \in \Gamma$  and  $q \in Q$ . In view of the definitions of the various actions,

$$\begin{aligned} \left( \beta_2(x^{-1}) (\beta_1(x) f) \right) (av_q) &= a^{x^{-1}} \left( (\beta_1(x) f) (v_q) \right) = a^{x^{-1}} \left( x f(x^{-1} v_q) \right) \\ &= a (i_x f i_x^{-1}) (v_q) \end{aligned}$$

since, with  $b = f(x^{-1} v_q) \in M_{e_Q}$ , in view of (vi),

$$x^{-1}(xb) = i_x(b).$$

(viii): The action of  $Q$  on  ${}_A \text{End}(M_{e_Q})$  in (ii) is given by the rule

$$({}^q f)(b) = x f(x^{-1} b), \quad f \in {}_A \text{End}(M_{e_Q}), \quad q \in Q,$$

where  $x \in \Gamma$  is a pre-image of  $q \in Q$ ; hence given  $u \in (A, Q, \varphi, \vartheta)$ ,

$$({}^q f_u)(b) = x f_u(x^{-1} b) = x x^{-1} b u = b u = f_u b.$$

On the other hand, let  $f: M_{e_Q} \rightarrow M_{e_Q}$  be an  $A$ -linear endomorphism of  $M_{e_Q}$  that is fixed under  $Q$ . Let  $q \in Q$  and choose a pre-image  $x \in \Gamma$  of  $q$ . Then, for any  $a \in A$ ,

$$f(av_q) = a f(v_q) = a x f(x^{-1} v_q) = a x^{-1} x v_q f(1) = a v_q f(1).$$

Hence  $f$  is given by multiplication in  $(A, Q, \varphi, \vartheta)$  by  $f(1)$ . The argument really comes down to the standard fact that, for any ring  $\Lambda$ , the algebra  ${}_{\Lambda} \text{End}(\Lambda)$  is canonically isomorphic to  $\Lambda^{\text{op}}$ .

(ix): This follows from Proposition 4.4(viii) since, by Galois descent (1.2) (ii), the canonical map  $S \otimes_R {}_A \text{End}(M_{e_Q})^Q \rightarrow {}_A \text{End}(M_{e_Q})$  is an isomorphism of  $S$ -algebras.

(xi): In view of Proposition 4.4(ix),  $S \otimes_R (A, Q, \varphi, \vartheta)$  is an Azumaya  $S$ -algebra, i.e., a central separable  $S$ -algebra. By [AG60a, Corollary A.5 p. 398], the ring  $S$  is separable over  $R$  and therefore, by [AG60a, Theorem 2.3 p. 374], the algebra  $S \otimes_R (A, Q, \varphi, \vartheta)$  is separable over  $R$ . By [AG60a, Proposition A.3], as an  $R$ -module, the ring  $R$  is a direct summand of  $S$  and hence, by [AG60a, Proposition 1.7],  $(A, Q, \varphi, \vartheta)$  is separable over  $R$ . By Proposition 4.3(iii) above, the ring  $R$  coincides with the center of  $(A, Q, \varphi, \vartheta)$ .  $\square$

*Remark 4.5.* Proposition 4.4(xi) is related with [Chi72, Corollary 5.2 p. 13] and with [Kan64, Theorem 4 p. 110].

*Remark 4.6.* In the special case where  $(A, \sigma)$  is the ground ring  $S$ , endowed with the given  $Q$ -action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ , the above crossed product construction is classical. In that case, when the group  $Q$  is finite, the assignment to a group extension  $e: U(S) \rightarrow \Gamma \xrightarrow{\pi^\Gamma} Q$  of its associated crossed product algebra  $(S, Q, e, \kappa_Q \circ \pi^\Gamma)$  yields the familiar homomorphism

$$H^2(Q, U(S)) \longrightarrow B(S|R) \quad (4.6)$$

into the subgroup  $B(S|R)$  of the Brauer group  $B(R)$  of  $R = S^Q$  that consists of the Brauer classes split by  $S$ .

## 5 Normal algebras with zero Teichmüller class

As before,  $S$  denotes a commutative ring and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of a group  $Q$  on  $S$ .

**Theorem 5.1.** *The Teichmüller class of a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  is zero if and only if, for  $I = Q$ , the  $Q$ -normal structure  $\sigma_I$  on the matrix algebra  $M_I(A)$  is equivariant.*

*Proof.* In view of Propositions 3.6 and 3.8, the condition is sufficient.

To show that the condition is necessary, consider a  $Q$ -normal algebra  $(A, \sigma)$  having zero Teichmüller class; then so has the opposite algebra  $(A^{\text{op}}, \sigma^{\text{op}})$ , by Proposition 3.7. Consider the crossed module  $(U(A^{\text{op}}), B^{\sigma^{\text{op}}}, \partial)$  which defines the crossed 2-fold extension  $e_{(A^{\text{op}}, \sigma^{\text{op}})}$  associated to  $(A^{\text{op}}, \sigma^{\text{op}})$ . By [Hue80, §10 Theorem], there is a group extension

$$e: 1 \longrightarrow U(A^{\text{op}}) \xrightarrow{j} \Gamma \longrightarrow Q \longrightarrow 1$$

together with a morphism

$$(1, \hat{\vartheta}): (U(A^{\text{op}}), \Gamma, j) \longrightarrow (U(A^{\text{op}}), B^{\sigma^{\text{op}}}, \partial)$$

of crossed modules inducing the identity map of  $Q$ . Via the canonical homomorphism from  $B^{\sigma^{\text{op}}}$  to  $\text{Aut}(A^{\text{op}})$ , the morphism  $(1, \hat{\vartheta})$  of crossed modules yields a morphism

$$(1, \vartheta): (U(A^{\text{op}}), \Gamma, j) \longrightarrow (U(A^{\text{op}}), \text{Aut}(A^{\text{op}}), \partial)$$

of crossed modules that induces  $\sigma^{\text{op}}$ . Then  $\text{End}_{A^{\text{op}}}(A^{\text{op}}, Q, e, \vartheta) = M_{|Q|}(A)$  and, by Proposition 4.4(vii), the  $Q$ -normal structure  $\sigma_{|Q|}: Q \rightarrow \text{Aut}(M_{|Q|}(A))$  is equivariant.  $\square$

When  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ , Theorem 5.1 has an obvious consequence to be phrased in terms of extensions of scalars, by Galois descent, cf. Propositions 3.6, 4.4(ix) and 4.4(x). We content ourselves with the following simplified form which in the classical case comes down to the corresponding result of Teichmüller [Tei40], see also Eilenberg-Mac Lane [EM48].

**Corollary 5.2.** *Suppose that  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ . Then the Teichmüller class of a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  is zero if and only if, for some matrix algebra  $M_I(A)$ , the  $Q$ -normal structure arises by scalar extension.*

*Remark 5.3.* In the corollary one cannot in general assert that, if the Teichmüller class of a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  is zero, the  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(A)$  itself comes from scalar extension, see [EM48, § 14]. Likewise, the  $Q$ -normal structure  $\sigma$  in Theorem 5.1 will not in general itself be  $Q$ -equivariant.

The Teichmüller class of a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  is the obstruction to the existence of a strong Deuring embedding relative to  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $A$  into an algebra  $C$  over  $R = S^Q$ , in the following sense:

**Theorem 5.4.** *The Teichmüller class of a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  is zero if and only if  $A$  admits a strong Deuring embedding*

$$(C, \chi: Q \rightarrow N^{\text{U}(C)}(A)/\text{U}(A))$$

*relative to  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  into an algebra  $C$  over  $R = S^Q$  that induces the  $Q$ -normal structure  $\sigma$  on  $A$  (in the sense that the associated homomorphism (3.11) coincides with  $\sigma$ , cf. Proposition 3.11(v)). Hence a central  $S$ -algebra  $A$  admits a strong Deuring embedding relative to  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  into some algebra  $C$  over  $R = S^Q$  if and only if  $A$  admits a  $Q$ -normal structure with zero Teichmüller class.*

*Proof.* Proposition 3.13 entails that a  $Q$ -normal structure arising from a strong Deuring embedding (relative to the structure map  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ ) has zero Teichmüller class.

Conversely, consider a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  with zero Teichmüller class. Since the Teichmüller class is zero, there is a group extension

$$e: 1 \longrightarrow \text{U}(A) \xrightarrow{j} \Gamma \xrightarrow{\pi} Q \longrightarrow 1, \quad (5.1)$$

cf. (4.1) above, together with a morphism

$$(1, \vartheta): (\text{U}(A), \Gamma, j) \longrightarrow (\text{U}(A), \text{Aut}(A), \partial) \quad (5.2)$$

of crossed modules that induces  $\sigma$ , cf. (4.2) above. The embedding of  $A$  into the crossed product algebra  $C = (A, Q, e, \vartheta)$ , cf. Proposition 4.3(i), together with the homomorphism  $\chi: Q \rightarrow N^{\text{U}(C)}(A)/\text{U}(A)$  induced by the injection  $\Gamma \rightarrow N^{\text{U}(C)}(A)$ , cf. Proposition 4.3(iv), is a strong Deuring embedding of  $A$  into  $C$  relative to  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  that induces the  $Q$ -normal structure  $\sigma$  on  $A$ .  $\square$

*Remark 5.5.* Under the circumstances of Theorem 5.4, since  $\Gamma$  injects into  $N^{\text{U}(C)}(A)$ , the structure map  $\chi: Q \rightarrow N^{\text{U}(C)}(A)/\text{U}(A)$  is injective as well.

Theorem 5.4 generalizes the corresponding results of Teichmüller [Tei40], in view of the following.



**Corollary 5.6.** *Let  $S|R$  be a Galois extension of commutative rings with Galois group  $Q$ , and let  $(A, \sigma)$  be a  $Q$ -normal  $S$ -algebra. Then the Teichmüller class of  $(A, \sigma)$  is zero if and only if there is a central  $R$ -algebra  $C$  which contains  $A$  in such a way that*

- (i) *the centralizer of  $S$  in  $C$  coincides with  $A$ , and*
- (ii) *each automorphism of  $S|R$ , i.e., each member  $q$  of  $Q$ , extends to an inner automorphism  $\alpha$  of  $C$  which (in view of (i)) maps  $A$  to itself, in such a way that the class of  $\alpha|A$  in  $\text{Out}(A)$  coincides with  $\sigma(q)$ . Moreover, if  $A$  is an Azumaya  $S$ -algebra then  $C$  may be taken to be an Azumaya  $R$ -algebra.*

*Proof.* The “if” part of the first assertion was given in Proposition 3.15. On the other hand, given a  $Q$ -normal  $S$ -algebra  $(A, \sigma)$  with zero Teichmüller class, the algebra  $C$  in the proof of Theorem 5.4 will have the desired properties, by Propositions 4.3 and 4.4(xi).  $\square$

In the classical case considered by Deuring [Deu36] and Teichmüller [Tei40] there was no need to spell out condition (ii) in Corollary 5.6, in view of the Skolem-Noether theorem.

The crossed product construction yields sort of a “generic” solution of the Deuring embedding problem, provided the solution exists, that is, the obstruction vanishes:

**Theorem 5.7.** *Let  $(A, \sigma)$  be a  $Q$ -normal  $S$ -algebra. Suppose that  $A$  admits a strong Deuring embedding  $(C, \chi: Q \rightarrow N^{U(C)}(A)/U(A))$  relative to the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$  into an algebra  $C$  over  $R = S^Q$  that induces the  $Q$ -normal structure  $\sigma$  on  $A$  in the sense that the composite of  $\chi$  with the homomorphism  $\eta_\sharp: N^{U(C)}(A)/U(A) \rightarrow \text{Out}(A)$  induced by conjugation in the normalizer  $N^{U(C)}(A)$  of  $A$  in the group  $U(C)$  of invertible elements of  $C$  coincides with the  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(A)$  on  $A$ , cf. Theorem 5.4 above. Then the strong Deuring embedding of  $A$  into  $C$  determines a group extension*

$$e: 1 \longrightarrow U(A) \xrightarrow{j} \Gamma \longrightarrow Q \longrightarrow 1,$$

*together with a morphism*

$$(1, \vartheta): (U(A), \Gamma, j) \longrightarrow (U(A), \text{Aut}(A), \partial)$$

*of crossed modules that induces  $\sigma$ , and the data induce a morphism*

$$(A, Q, e, \vartheta) \longrightarrow C \tag{5.3}$$

*of algebras over  $R = S^Q$  which is compatible with the strong Deuring embeddings.*

*Proof.* Let  $e$  be the group extension (3.12) and  $(1, \vartheta)$  the morphism (3.13) of crossed modules. Recall that  $N^{U(C)}(A)$  denotes the normalizer of  $A$  in the group  $U(C)$  of invertible elements of  $C$ . By construction, the group  $\Gamma$  is the fiber product group  $\Gamma = N^{U(C)}(A) \times_{N^{U(C)}(A)/U(A)} Q$ , the requisite homomorphism from  $Q$  to  $N^{U(C)}(A)/U(A)$  being the strong Deuring embedding structure map  $\chi: Q \rightarrow N^{U(C)}(A)/U(A)$ .

The canonical morphism  $A^t \Gamma \rightarrow C$  of algebras induces the morphism  $(A, Q, e, \vartheta) \rightarrow C$  of algebras we seek. In particular, this morphism of algebras restricts to a homomorphism  $U(A, Q, e, \vartheta) \rightarrow U(C)$  between the groups of invertible elements, and this homomorphism, in turn, restricts to a homomorphism

$$N^{U(A, Q, e, \vartheta)}(A) \longrightarrow N^{U(C)}(A)$$

from the normalizer  $N^{U(A, Q, e, \vartheta)}(A)$  of  $A$  in  $U(A, Q, e, \vartheta)$  to the normalizer  $N^{U(C)}(A)$  of  $A$  in  $U(C)$ . The composite

$$Q \longrightarrow N^{U(A, Q, e, \vartheta)}(A)/U(A) \longrightarrow N^{U(C)}(A)/U(A) \quad (5.4)$$

of the strong Deuring embedding structure map with respect to  $(A, Q, e, \vartheta)$  and the induced homomorphism  $N^{U(A, Q, e, \vartheta)}(A)/U(A) \rightarrow N^{U(C)}(A)/U(A)$  yields the strong Deuring embedding structure map  $\chi$ .  $\square$

## 6 Normal ring extensions

In the rest of the paper, we need a construction of certain normal and that of certain equivariant algebras involving the concept of a “normal Galois extension” in a sense we now explain.

As before,  $S$  denotes a commutative ring and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of a group  $Q$  on  $S$ . Let  $T|S$  be a Galois extension of commutative rings with Galois group  $N = \text{Aut}(T|S)$ . We shall refer to  $T|S$  as being  $Q$ -normal when each automorphism  $\kappa_Q(q)$  of  $S$ , as  $q$  ranges over  $Q$ , extends to an automorphism of  $T$ .

Somewhat more formally, given a Galois extension  $T|S$  of commutative rings with Galois group  $N$ , denote by  $\text{Aut}^S(T)$  the group of those automorphisms of  $T$  that map  $S$  to itself, let  $\text{res}: \text{Aut}^S(T) \rightarrow \text{Aut}(S)$  denote the obvious restriction map, so that  $N = \text{Aut}(T|S)$  is the kernel of  $\text{res}$ , let  $G$  denote the fiber product group  $G = \text{Aut}^S(T) \times_{\text{Aut}(S)} Q$  relative to  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ , and let  $\pi_Q: G \rightarrow Q$  denote the canonical homomorphism and  $i^N: N \rightarrow G$  the obvious injection. The obvious homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$  makes the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \xrightarrow{i^N} & G & \xrightarrow{\pi_Q} & Q \\ & & \parallel & & \kappa_G \downarrow & & \kappa_Q \downarrow \\ 1 & \longrightarrow & \text{Aut}(T|S) & \longrightarrow & \text{Aut}^S(T) & \xrightarrow{\text{res}} & \text{Aut}(S) \end{array} \quad (6.1)$$

commutative, where the unlabeled arrow is the obvious homomorphism. This diagram is a special case of a diagram of the kind (2.19). The Galois extension  $T|S$  of commutative rings is plainly  $Q$ -normal if and only if the homomorphism  $\pi_Q: G \rightarrow Q$  is surjective, that is, if and only if the sequence

$$e_{(T|S)}: 1 \longrightarrow N \xrightarrow{i^N} G \xrightarrow{\pi_Q} Q \longrightarrow 1 \quad (6.2)$$

is exact, i.e., an extension of  $Q$  by  $N$ . Given a  $Q$ -normal Galois extension  $T|S$  of commutative rings, we refer to the corresponding group extension (6.2) as the *associated structure extension* and to the corresponding homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$  as the *associated structure homomorphism*. It is immediate that a  $Q$ -normal Galois extension  $T|S$  with structure extension (6.2) and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ , the injection  $S \subseteq T$  being denoted by  $i: S \subseteq T$ , yields the morphism

$$(i, \pi_Q): (S, Q, \kappa_Q) \longrightarrow (T, G, \kappa_G) \quad (6.3)$$

in the change of actions category *Change* introduced in Subsection 2.7 above.

**Example 6.1.** Let  $K|P$  be a Galois extension of algebraic number fields, with  $G = \text{Gal}(K|P)$ . Let  $Z$  be a subfield of  $K$  that contains  $P$  and is a normal extension of  $P$ , and let  $N = \text{Gal}(K|Z)$

and  $Q = \text{Gal}(Z|P)$ . Let  $T$ ,  $S$  and  $R$  denote the rings of integers in, respectively,  $K$ ,  $Z$  and  $P$ . Suppose that  $K|Z$  is unramified but that  $Z|P$  is ramified. Then  $T|S$  is a  $Q$ -normal Galois extension of commutative rings but  $T|R$  and  $S|R$  are not Galois extensions of commutative rings.

Let  $(S, Q, \kappa)$  and  $(\hat{S}, \hat{Q}, \hat{\kappa})$  be objects of the change of actions category *Change* introduced in Subsection 2.7, and let  $T|S$  and  $\hat{T}|\hat{S}$  be normal Galois extension of commutative rings with respect to  $Q$  and  $\hat{Q}$ , with structure extensions  $e_{(T|S)}: N \rightarrowtail G \twoheadrightarrow Q$  and  $e_{(\hat{T}|\hat{S})}: \hat{N} \rightarrowtail \hat{G} \twoheadrightarrow \hat{Q}$  and structure homomorphisms  $\kappa_G: G \rightarrow \text{Aut}^S(T)$  and  $\kappa_{\hat{G}}: \hat{G} \rightarrow \text{Aut}^{\hat{S}}(\hat{T})$ , respectively. Then a *morphism*

$$(h, \phi): T|S \longrightarrow \hat{T}|\hat{S}$$

of normal Galois extensions consists of a ring homomorphism  $h: T \rightarrow \hat{T}$  and a group homomorphism  $\phi: \hat{G} \rightarrow G$  such that

- (i)  $f = h|S$  is a ring homomorphism  $S \rightarrow \hat{S}$ ,
- (ii) the values of  $\phi|\hat{N}$  lie in  $N$ , that is,  $\phi|\hat{N}$  is a homomorphism  $\hat{N} \rightarrow N$ , and
- (iii)  $h(\phi(\hat{x})t) = \hat{x}(h(t))$ ,  $\hat{x} \in \hat{G}$ ,  $t \in T$ .

## 7 Crossed pair algebras

As before,  $S$  denotes a commutative ring and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of a group  $Q$  on  $S$ . In this section we will use the results of [Hue81b] to offer a partial answer to the question as to which classes in  $H^3(Q, U(S))$  are Teichmüller classes. Our result extends the classical answer of Eilenberg and Mac Lane [EM48] (reproduced in [HS53]); later in the paper we shall give a complete answer.

### 7.1 Crossed pairs

For intelligibility, we recall that notion from [Hue81b, p. 152].

Let

$$1 \longrightarrow N \xrightarrow{i^N} G \longrightarrow Q \longrightarrow 1 \quad (7.1)$$

be a group extension and  $M$  a  $G$ -module; we write the  $G$ -action  $G \times M \longrightarrow M$  on  $M$  as  $(x, y) \mapsto {}^x y$ , for  $x \in G$  and  $y \in M$ . Further, let  $e: M \rightarrowtail \Gamma \xrightarrow{\pi_N} N$  be a group extension whose class  $[e] \in H^2(N, M)$  is fixed under the standard  $Q$ -action on  $H^2(N, M)$ . Given  $x \in G$ , we write

$$\ell_x(y) = {}^x y, \quad y \in M, \quad i_x(n) = x n x^{-1}, \quad n \in N.$$

Write  $\text{Aut}_G(e)$  for the subgroup of  $\text{Aut}(\Gamma) \times G$  that consists of those pairs  $(\alpha, x)$  which make the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & \Gamma & \longrightarrow & N \longrightarrow 1 \\ & & \ell_x \downarrow & & \alpha \downarrow & & i_x \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & \Gamma & \longrightarrow & N \longrightarrow 1 \end{array}$$

commutative.

The homomorphism

$$\beta: \Gamma \longrightarrow \text{Aut}_G(e), \quad \beta(y) = (i_y, i^N(\pi_N(y))), \quad y \in \Gamma,$$

together with the obvious action of  $\text{Aut}_G(e)$  on  $\Gamma$ , yields a crossed module  $(\Gamma, \text{Aut}_G(e), \beta)$  whence, in particular,  $\beta(\Gamma)$  is a normal subgroup of  $\text{Aut}_G(e)$ ; with the notation  $\text{Out}_G(e)$  for the cokernel of  $\beta$ , the resulting crossed 2-fold extension has the form

$$\hat{e}: 0 \longrightarrow M^N \longrightarrow \Gamma \xrightarrow{\beta} \text{Aut}_G(e) \longrightarrow \text{Out}_G(e) \longrightarrow 1. \quad (7.2)$$

The map  $\text{Der}(N, M) \longrightarrow \text{Aut}_G(e)$  given by the association

$$\text{Der}(N, M) \ni d \longmapsto (\alpha_d, 1), \quad \alpha_d(y) = (d\pi_N(y))y, \quad y \in \Gamma,$$

is an injective homomorphism; this homomorphism and the obvious map  $\text{Aut}_G(e) \rightarrow G$  yield the group extension

$$0 \longrightarrow \text{Der}(N, M) \longrightarrow \text{Aut}_G(e) \longrightarrow G \longrightarrow 1,$$

the map  $\text{Aut}_G(e) \rightarrow G$  being surjective, since the class  $[e] \in H^2(N, M)$  is supposed to be fixed under  $Q$ . Further, let  $\zeta: M \rightarrow \text{Der}(N, M)$  be the homomorphism defined by  $(\zeta(m))(n) = m(nm)^{-1}$ , as  $m$  ranges over  $M$  and  $n$  over  $N$ . With these preparations out of the way, the data fit into the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & M^N & \xlongequal{\quad} & M^N & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & \Gamma & \xrightarrow{\pi_N} & N \longrightarrow 1 \\ & & \zeta \downarrow & & \beta \downarrow & & i^N \downarrow \\ 0 & \longrightarrow & \text{Der}(N, M) & \longrightarrow & \text{Aut}_G(e) & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(N, M) & \longrightarrow & \text{Out}_G(e) & \longrightarrow & Q \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array} \quad (7.3)$$

with exact rows and columns. We shall use the notation

$$\bar{e}: 0 \longrightarrow H^1(N, M) \longrightarrow \text{Out}_G(e) \longrightarrow Q \longrightarrow 1$$

for the bottom row extension of (7.3). This extension is the cokernel, in the category of group extensions with abelian kernel, of the morphism  $(\zeta, \beta, i)$  of group extensions.

Suppose now that the extension  $\bar{e}$  splits; we will then say that  $e$  *admits a crossed pair structure*, and we will refer to a section  $\psi: Q \rightarrow \text{Out}_G(e)$  of  $\bar{e}$  as a *crossed pair structure on the group extension*  $e: M \rightarrow \Gamma \xrightarrow{\pi_N} N$  with respect to the group extension (7.1). By definition, a *crossed pair*  $(e, \psi)$  with respect to the group extension (7.1) and the  $G$ -module  $M$  consists of a group extension  $e: M \rightarrow \Gamma \rightarrow N$  whose class  $[e] \in H^2(N, M)$  is fixed under  $Q$  such that the associated extension  $\bar{e}$  splits, together with a section  $\psi: Q \rightarrow \text{Out}_G(e)$  of  $\bar{e}$  [Hue81b, p. 152].

Suitable classes of crossed pairs with respect to (7.1) and the  $G$ -module  $M$  constitute an abelian group  $\text{Xpext}(G, N; M)$  [Hue81b, Theorem 1]. Moreover, cf. [Hue81b, Theorem 2], suitably defined homomorphisms

$$j: H^2(G, M) \longrightarrow \text{Xpext}(G, N; M), \quad \Delta: \text{Xpext}(G, N; M) \longrightarrow H^3(Q, M^N)$$

yield an extension of the classical five term exact sequence to an eight term exact sequence of the kind

$$\begin{aligned} 0 \longrightarrow H^1(Q, M^N) &\xrightarrow{\text{inf}} H^1(G, M) \xrightarrow{\text{res}} H^1(N, M)^Q \xrightarrow{\Delta} H^2(Q, M^N) \\ &\xrightarrow{\text{inf}} H^2(G, M) \xrightarrow{j} \text{Xpext}(G, N; M) \xrightarrow{\Delta} H^3(Q, M^N) \xrightarrow{\text{inf}} H^3(G, M). \end{aligned} \quad (7.4)$$

For later reference, we recall the construction of  $\Delta$ . To this end, given a crossed pair

$$(e: 0 \rightarrow M \rightarrow \Gamma \rightarrow N \rightarrow 1, \quad \psi: Q \rightarrow \text{Out}_G(e))$$

with respect to the group extension (7.1) and the  $G$ -module  $M$ , let  $B^\psi$  denote the fiber product group  $\text{Aut}_G(e) \times_{\text{Out}_G(e)} Q$  with respect to the crossed pair structure map  $\psi: Q \rightarrow \text{Out}_G(e)$  and, furthermore, let  $\partial^\psi: \Gamma \rightarrow B^\psi$  denote the obvious homomorphism; together with the obvious action of  $B^\psi$  on  $\Gamma$  induced by the canonical homomorphism  $B^\psi \rightarrow \text{Aut}_G(e)$ , the exact sequence

$$e_\psi: 0 \longrightarrow M^N \longrightarrow \Gamma \xrightarrow{\partial^\psi} B^\psi \longrightarrow Q \longrightarrow 1 \quad (7.5)$$

is a crossed 2-fold extension and hence represents a class in  $H^3(Q, M^N)$ . We shall refer to  $e_\psi$  as the *crossed 2-fold extension associated to the crossed pair*  $(e, \psi)$ . The homomorphism  $\Delta: \text{Xpext}(G, N; M) \rightarrow H^3(Q, M^N)$  is given by the assignment to a crossed pair  $(e, \psi)$  of its associated crossed 2-fold extension  $e_\psi$ .

*Remark 7.1.* By [Hue81a, Theorem 1], the association  $e \mapsto \bar{e}$  yields a conceptual description of the differential  $d_2: E_2^{0,2} \rightarrow E_2^{2,1}$  of the Lyndon-Hochschild-Serre spectral sequence  $(E_r^{p,q}, d_r)$  associated to the group extension (7.1) and the  $G$ -module  $M$ .

**Proposition 7.2.** *In the special case where the  $N$ -action on  $M$  is trivial, given a group extension  $e: M \rightarrow \Gamma \xrightarrow{\pi_N} N$  that admits a crossed pair structure, crossed pair structures  $\psi: Q \rightarrow \text{Out}_G(e)$  on the group extension  $e$  correspond bijectively to actions of  $G$  on  $\Gamma$  that turn  $i^N \circ \pi_N: \Gamma \rightarrow G$  into a crossed module in such a way that the canonical homomorphism  $G \rightarrow B^\psi = \text{Aut}_G(e) \times_{\text{Out}_G(e)} Q$  is an isomorphism.  $\square$*

*Remark 7.3.* Given the group extension (7.1), consider a group extension

$$e: 1 \longrightarrow X \longrightarrow K \xrightarrow{\pi_N} N \longrightarrow 1,$$

the group  $X$  not necessarily being abelian, let  $\phi = i^N \circ \pi_N: K \rightarrow G$  denote the composite of  $i$  and  $\pi_N$ , and let  $\text{Aut}(e)$  denote the subgroup of  $\text{Aut}(K)$  that consists of the automorphisms of  $K$  that map  $X$  to itself; such a homomorphism  $\phi$  is referred to in [Tay53] as a *normal homomorphism*. Conjugation in  $K$  yields a homomorphism  $\beta: K \rightarrow \text{Aut}(e)$  from  $K$  onto a normal subgroup  $\beta(K)$  of  $\text{Aut}(e)$ , and the restriction  $\zeta$  of  $\beta$  to  $X$ , that is, conjugation in  $K$  with elements of  $X$ , yields a homomorphism  $\zeta: X \rightarrow \text{Aut}(e)$  from  $X$  onto a normal subgroup

$\zeta(X)$  of  $\text{Aut}(e)$  as well; let  $\text{can}: \text{Aut}(e) \rightarrow \text{Aut}(e)/\zeta(X)$  denote the canonical surjection. A *modular structure on  $\phi$*  is a homomorphism  $\theta: G \rightarrow \text{Aut}(e)/\zeta(X)$  making the diagram

$$\begin{array}{ccc} K & \xrightarrow{\pi_N} & N \\ \downarrow \beta & \searrow \phi & \downarrow i^N \\ \text{Aut}(e) & & G \\ \downarrow \text{can} & \swarrow \theta & \\ \text{Aut}(e)/\zeta(X) & & \end{array}$$

commutative [Tay53]. A *pseudo-module* is defined to be a pair  $(\phi, \theta)$  that consists of a normal homomorphism  $\phi$  and a modular structure  $\theta$  on  $\phi$  [Tay53].

Let  $(\phi, \theta)$  be a pseudo-module and consider the two abstract kernels  $G \rightarrow \text{Out}(X)$  and  $Q \rightarrow \text{Out}(K)$  induced by that pseudo-module. Now, fix an abstract  $G$ -kernel structure  $\omega: G \rightarrow \text{Out}(X)$  on  $X$  in advance and consider the group  $\text{Aut}_G(e)$  that consists of the pairs  $(\alpha, x) \subseteq \text{Aut}(e) \times G$  which make the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & X & \longrightarrow & K & \longrightarrow & N \longrightarrow 1 \\ & & \alpha|X \downarrow & & \alpha \downarrow & & i_x \downarrow \\ 1 & \longrightarrow & X & \longrightarrow & K & \longrightarrow & N \longrightarrow 1 \end{array}$$

commutative in such a way that the image of  $\alpha|X$  in  $\text{Out}(X)$  coincides with  $\omega(x) \in \text{Out}(X)$ . Then the modular structures on  $\phi$  that induce, in particular, the abstract  $G$ -kernel structure  $\omega$  on  $X$  are given by homomorphisms  $\theta: G \rightarrow \text{Aut}_G(e)/\zeta(X)$ . In the special case where  $X$  is abelian, an abstract  $G$ -kernel structure on  $X$  is an ordinary  $G$ -module structure, and those modular structures  $\theta: G \rightarrow \text{Aut}_G(e)/\zeta(X)$  correspond bijectively to crossed pair structures  $\psi: Q \rightarrow \text{Out}_G(e)$  on  $e$ .

## 7.2 Crossed pairs and normal algebras

Let  $T|S$  be a  $Q$ -normal Galois extension of commutative rings, with structure extension

$$e_{(T|S)}: 1 \longrightarrow N \xrightarrow{i^N} G \longrightarrow Q \longrightarrow 1$$

and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ ; in particular, the group  $N$  is finite. Let  $(e: U(T) \twoheadrightarrow \Gamma \twoheadrightarrow N, \psi: Q \rightarrow \text{Out}_G(e))$  be a crossed pair with respect to the group extension  $e_{(T|S)}$  and the  $G$ -module  $U(T)$ . The corresponding crossed 2-fold extension (7.2) now takes the form

$$\hat{e}: 0 \longrightarrow U(S) \longrightarrow \Gamma \longrightarrow \text{Aut}_G(e) \longrightarrow \text{Out}_G(e) \longrightarrow 1.$$

To the crossed pair  $(e, \psi)$ , we shall associate a  $Q$ -normal  $S$ -algebra  $(A_e, \sigma_\psi)$  as follows.

The composite  $\vartheta: \Gamma \rightarrow N \rightarrow \text{Aut}(T)$  yields an action of  $\Gamma$  on  $T$ ; let  $A_e$  denote the crossed product algebra  $(T, N, e, \vartheta)$ . Since the group  $N$  is finite,  $A_e$  is an Azumaya  $S$ -algebra; this fact also follows from Proposition 4.4(xi). Recall that there is an obvious injection  $i: \Gamma \rightarrow U(A_e)$ . The following is immediate.

**Proposition 7.4.** *The rule*

$$i_{\sharp}^{(\alpha, x)}(ty) = ({}^xt)({}^{\alpha}y), \quad (7.6)$$

as  $t$  ranges over  $T$ ,  $y$  over  $\Gamma$ , and  $(\alpha, x)$  over  $\text{Aut}_G(e)$  ( $\subseteq \text{Aut}(\Gamma) \times G$ ), yields a morphism

$$(i, i_{\sharp}): (\Gamma, \text{Aut}_G(e), \beta) \longrightarrow (\text{U}(A_e), \text{Aut}(A_e, Q), \partial)$$

of crossed modules which, in turn, induces the morphism

$$\begin{array}{ccccccccc} \hat{e}: 0 & \longrightarrow & \text{U}(S) & \longrightarrow & \Gamma & \xrightarrow{\beta} & \text{Aut}_G(e) & \longrightarrow & \text{Out}_G(e) & \longrightarrow & 1 \\ & & \parallel & & i \downarrow & & i_{\sharp} \downarrow & & i_b \downarrow & & \\ e_{(A_e, Q)}: 0 & \longrightarrow & \text{U}(S) & \longrightarrow & \text{U}(A_e) & \xrightarrow{\partial} & \text{Aut}(A_e, Q) & \longrightarrow & \text{Out}(A_e, Q) & \longrightarrow & 1 \end{array}$$

of crossed 2-fold extensions, where  $i_b$  denotes the induced homomorphism.

Given a crossed pair  $(e: 0 \rightarrow \text{U}(T) \rightarrow \Gamma \rightarrow N \rightarrow 1, \psi: Q \rightarrow \text{Out}_G(e))$  with respect to the group extension  $e_{(T|S)}$  and the  $G$ -module  $\text{U}(T)$ , let

$$\sigma_{\psi} = i_b \circ \psi: Q \longrightarrow \text{Out}_G(e) \longrightarrow \text{Out}(A_e, Q);$$

it is then obvious that  $(A_e, \sigma_{\psi})$  is a  $Q$ -normal (Azumaya)  $S$ -algebra, and we will refer to  $(A_e, \sigma_{\psi})$  as a  $Q$ -normal crossed pair algebra with respect to the  $Q$ -normal Galois extension  $T|S$  of commutative rings.

**Theorem 7.5.** *Let  $T|S$  be a  $Q$ -normal Galois extension of commutative rings, with structure extension  $e_{(T|S)}: N \rightarrow G \rightarrow Q$  and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ , cf. Section 6 above. Then a class  $k \in \text{H}^3(Q, \text{U}(S))$  is the Teichmüller class of some crossed pair algebra  $(A_e, \sigma_{\psi})$  with respect to the  $Q$ -normal Galois extension  $T|S$  if and only if  $k$  is split in  $T|S$  in the sense that  $k$  goes to zero under inflation  $\text{H}^3(Q, \text{U}(S)) \rightarrow \text{H}^3(G, \text{U}(T))$ .*

With  $M = \text{U}(T)$  and  $M^N = \text{U}(S)$ , the theorem is a consequence of the exactness, at  $\text{H}^3(Q, \text{U}(S))$ , of the sequence (7.4). Indeed, by construction, the homomorphism  $\Delta$  is given by the assignment to a crossed pair  $(e: \text{U}(T) \rightarrow \Gamma \rightarrow N, \psi: Q \rightarrow \text{Out}_G(e))$  with respect to the group extension  $e_{(T|S)}$  and the  $G$ -module  $\text{U}(T)$  of the corresponding crossed 2-fold extension (7.5), which now takes the form

$$e_{\psi}: 0 \longrightarrow \text{U}(S) \longrightarrow \Gamma \xrightarrow{\partial^{\psi}} B^{\psi} \longrightarrow Q \longrightarrow 1.$$

Theorem 7.5 is therefore a consequence of the following, which is again immediate.

**Proposition 7.6.** *Given a crossed pair  $(e, \psi)$  with respect to the group extension  $e_{(T|S)}$  and the  $G$ -module  $\text{U}(T)$ , the morphism  $(i, i_{\sharp})$  of crossed modules in Proposition 7.4 above induces a congruence morphism*

$$\begin{array}{ccccccccc} e_{\psi}: 0 & \longrightarrow & \text{U}(S) & \longrightarrow & \Gamma & \xrightarrow{\partial^{\psi}} & B^{\psi} & \longrightarrow & Q & \longrightarrow & 1 \\ & & \parallel & & i \downarrow & & \hat{i} \downarrow & & \parallel & & \\ e_{(A_e, \sigma_{\psi})}: 0 & \longrightarrow & \text{U}(S) & \longrightarrow & \text{U}(A_e) & \xrightarrow{\partial^{\sigma_{\psi}}} & B^{\sigma_{\psi}} & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

of crossed 2-fold extensions.

*Proof of Theorem 7.5.* By exactness, it is immediate that the Teichmüller class of any crossed pair algebra  $(A_e, \sigma_\psi)$  with respect to  $T|S$  is split in  $T|S$ . Hence the condition is necessary. To establish sufficiency, consider a class  $k \in H^3(Q, U(S))$  which is split in  $T|S$ , that is, goes to zero under inflation  $H^3(Q, U(S)) \rightarrow H^3(G, U(T))$ . By exactness,  $k$  then arises from some crossed pair  $(e, \psi)$  with respect to the group extension  $e_{(T|S)}$  and the  $G$ -module  $U(T)$ , that is,

$$k = [e_\psi] \in H^3(Q, U(S)).$$

By Proposition 7.6, the Teichmüller class of the associated crossed pair algebra  $(A_e, \sigma_\psi)$  with respect to  $T|S$  coincides with  $[e_\psi] = k$ .  $\square$

## 8 Normal Deuring embedding and Galois descent for Teichmüller classes

As before,  $S$  denotes a commutative ring and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of a group  $Q$  on  $S$ . Let  $T|S$  be a  $Q$ -normal Galois extension of commutative rings, with structure extension

$$e_{(T|S)}: 1 \longrightarrow N \xrightarrow{i^N} G \xrightarrow{\pi_Q} Q \longrightarrow 1 \quad (8.1)$$

and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ , cf. (6.1). In this section, we will prove, among others, that if a class  $k \in H^3(Q, U(S))$  goes under inflation to the Teichmüller class in  $H^3(G, U(T))$  of some  $G$ -normal  $T$ -algebra, then  $k$  is itself the Teichmüller class of some  $Q$ -normal  $S$ -algebra. To this end, we will reexamine Deuring's embedding problem, cf. Subsection 3.9 and Section 5 above

### 8.1 The definitions

Let  $A$  be a central  $T$ -algebra,  $(C, \sigma_Q: Q \rightarrow \text{Out}(C))$  a  $Q$ -normal  $S$ -algebra, and  $A \subseteq C$  an embedding of  $A$  into  $C$ . We shall refer to the embedding of  $A$  into  $C$  as a  *$Q$ -normal Deuring embedding with respect to  $\sigma_Q: Q \rightarrow \text{Out}(C)$*  and (8.1) if each automorphism  $\kappa_G(x)$  of  $T$ , as  $x$  ranges over  $G$ , extends to an automorphism  $\alpha$  of  $C$  in such a way that

- (i)  $[\alpha] = \sigma_Q(\pi_Q(x)) \in \text{Out}(C)$ , and
- (ii)  $\alpha$  maps  $A$  to itself.

*Remark 8.1.* In the special case where  $Q$  is the trivial group, the group  $G$  boils down to the group  $N = \text{Aut}(S|R)$  and, since each automorphism  $\alpha$  of  $C$  that extends some  $x \in N$  is required to map  $A$  to itself and to map to the trivial element of  $\text{Out}(C)$ , that automorphism  $\alpha$  necessarily extends to an inner automorphism of  $C$  that normalizes  $A$ ; thus the notion of normal Deuring embedding then comes down to the notion of Deuring embedding introduced in Subsection 3.9 above.

*Remark 8.2.* Given an embedding of  $A$  into  $C$  such that  $A$  coincides with the centralizer of  $T$  in  $C$ , an automorphism  $\alpha$  of  $C$  extending an automorphism  $\kappa_G(x)$  of  $T$  for  $x \in G$  necessarily maps  $A$  to itself. Thus, in the definition of a  $Q$ -normal Deuring embedding, condition (ii) is then redundant.



For technical reasons, we need a stronger concept of a normal Deuring embedding. We will now prepare for this definition.

Let  $A$  be a central  $T$ -algebra,  $C$  a central  $S$ -algebra, and suppose the algebra  $A$  to be embedded into  $C$ . Recall the crossed module  $(U(C), \text{Aut}(C), \partial_C)$  associated to the central  $S$ -algebra  $C$ , and consider the associated crossed 2-fold extension

$$e_C: 0 \longrightarrow U(S) \longrightarrow U(C) \xrightarrow{\partial_C} \text{Aut}(C) \longrightarrow \text{Out}(C) \longrightarrow 1, \quad (8.2)$$

cf. (3.1). The normalizer  $N^{U(C)}(A)$  of  $A$  in  $U(C)$  and the centralizer  $C^{U(C)}(T)$  of  $T$  in  $U(C)$ , together with  $U(A)$  and  $U(C)$ , constitute an ascending sequence

$$U(A) \subseteq C^{U(C)}(T) \subseteq N^{U(C)}(A) \subseteq U(C)$$

of groups. When  $A$  coincides with the centralizer of  $T$  in  $C$ , the inclusion  $U(A) \subseteq C^{U(C)}(T)$  is the identity.

We continue with the general case where  $A$  does not necessarily coincide with the centralizer of  $T$  in  $C$ . Let  $\text{Aut}^A(C)$  denote the group of automorphisms of  $C$  that map  $A$  to itself. The action of  $\text{Aut}(C)$  on  $U(C)$  induces an action of  $\text{Aut}^A(C)$  on each of the groups  $U(A)$ ,  $C^{U(C)}(T)$ , and  $N^{U(C)}(A)$ , and the restrictions of the homomorphism  $\partial_C$  together with the actions yield three crossed modules

$$(N^{U(C)}(A), \text{Aut}^A(C), \partial_C^N) \quad (8.3)$$

$$(C^{U(C)}(T), \text{Aut}^A(C), \partial_C^T) \quad (8.4)$$

$$(U(A), \text{Aut}^A(C), \partial_C^A), \quad (8.5)$$

each homomorphism  $\partial_C^N$ ,  $\partial_C^T$ ,  $\partial_C^A$  being the corresponding restriction of the homomorphism  $\partial_C: U(C) \rightarrow \text{Aut}(C)$ . We write the associated crossed 2-fold extensions as

$$e_C^A: 0 \longrightarrow U(S) \longrightarrow U(A) \xrightarrow{\partial_C^A} \text{Aut}^A(C) \longrightarrow \text{Out}(C, A) \longrightarrow 1 \quad (8.6)$$

$$e_C^T: 0 \longrightarrow U(S) \longrightarrow C^{U(C)}(T) \xrightarrow{\partial_C^T} \text{Aut}^A(C) \longrightarrow \text{Out}(C, T) \longrightarrow 1 \quad (8.7)$$

$$e_C^N: 0 \longrightarrow U(S) \longrightarrow N^{U(C)}(A) \xrightarrow{\partial_C^N} \text{Aut}^A(C) \longrightarrow \text{Out}^A(C) \longrightarrow 1, \quad (8.8)$$

the groups  $\text{Out}(C, A)$ ,  $\text{Out}(C, T)$ , and  $\text{Out}^A(C)$  being defined by exactness. The inclusions  $U(A) \subseteq C^{U(C)}(T) \subseteq N^{U(C)}(A)$  induce a commutative diagram

$$\begin{array}{ccccccccc} e_C^A: 0 & \longrightarrow & U(S) & \longrightarrow & U(A) & \xrightarrow{\partial_C^A} & \text{Aut}^A(C) & \longrightarrow & \text{Out}(C, A) & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \parallel & & \downarrow & & \\ e_C^T: 0 & \longrightarrow & U(S) & \longrightarrow & C^{U(C)}(T) & \xrightarrow{\partial_C^T} & \text{Aut}^A(C) & \longrightarrow & \text{Out}(C, T) & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \parallel & & \downarrow & & \\ e_C^N: 0 & \longrightarrow & U(S) & \longrightarrow & N^{U(C)}(A) & \xrightarrow{\partial_C^N} & \text{Aut}^A(C) & \longrightarrow & \text{Out}^A(C) & \longrightarrow & 1 \end{array}$$

of morphisms of crossed 2-fold extensions and, by diagram chase, the induced homomorphisms  $\text{Out}(C, A) \rightarrow \text{Out}(C, T)$  and  $\text{Out}(C, T) \rightarrow \text{Out}^A(C)$  are surjective.

Restriction induces canonical homomorphisms

$$\text{res}: \text{Out}(C, A) \longrightarrow \text{Out}^S(A), \text{ res}: \text{Out}(C, T) \longrightarrow \text{Aut}^S(T)$$

(where the notation “res” is slightly abused) in such a way that the diagram

$$\begin{array}{ccc} \text{Out}(C, A) & \xrightarrow{\text{res}} & \text{Out}^S(A) \\ \downarrow & & \downarrow \text{res} \\ \text{Out}(C, T) & \xrightarrow{\text{res}} & \text{Aut}^S(T) \end{array}$$

is commutative. Moreover, the obvious homomorphism  $\text{Out}^A(C) \rightarrow \text{Out}(C)$  is injective, and we shall identify  $\text{Out}^A(C)$  with its isomorphic image in  $\text{Out}(C)$  if need be.

Now, given a homomorphism  $\chi_G: G \rightarrow \text{Out}(C, A)$ , its composite with the restriction map  $\text{res}: \text{Out}(C, A) \rightarrow \text{Out}(A)$  yields a  $G$ -normal structure on  $A$ . However, in order for such a homomorphism to match the other data, in particular the given  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$ , we must impose further conditions. We now spell out the details.

Let  $\partial_{C,\#}^A: U(A)/U(S) \rightarrow \text{Aut}^A(C)$  denote the (injective) homomorphism induced by the crossed module structure map  $\partial_C^A$  in the crossed module (8.5). The crossed modules (8.3) and (8.5) yield the commutative diagram

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U(S) & \longrightarrow & U(A) & \longrightarrow & U(A)/U(S) \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \partial_{C,\#}^A \\ 0 & \longrightarrow & U(S) & \longrightarrow & N^{U(C)}(T) & \xrightarrow{\partial_C^N} & \text{Aut}^A(C) \longrightarrow \text{Out}^A(C) \longrightarrow 1 \\ & & & & \downarrow & & \downarrow \\ & & & & 1 \longrightarrow N^{U(C)}(A)/U(A) & \longrightarrow & \text{Out}(C, A) \longrightarrow \text{Out}^A(C) \longrightarrow 1 \\ & & & & \downarrow & & \downarrow \\ & & & & 1 & & 1 \end{array}$$

with exact rows and columns, the third row being defined by exactness. This third row is an ordinary group extension, and we shall write it as

$$e_{(A,C)}: 1 \longrightarrow N^{U(C)}(A)/U(A) \longrightarrow \text{Out}(C, A) \longrightarrow \text{Out}^A(C) \longrightarrow 1. \quad (8.9)$$

We define a *strong  $Q$ -normal Deuring embedding* of  $A$  into  $C$  with respect to the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$  and to the structure extension (8.1) to consist of an embedding of  $A$  into  $C$  together with a homomorphism  $\chi_G: G \rightarrow \text{Out}(C, A)$  that is compatible with the other data in the following sense:

- The restriction  $\chi_N: N \rightarrow N^{U(C)}(A)/U(A)$  to  $N = \text{Aut}(T|S)$  of the homomorphism  $\chi_G$  turns the embedding of  $A$  into  $C$  into a strong Deuring embedding relative to the action

$\text{Id}: N \rightarrow \text{Aut}(T|S)$  of  $N$  on  $T$  in such a way that the diagram

$$\begin{array}{ccccccc}
e_{(T|S)}: 1 & \longrightarrow & N & \xrightarrow{i^N} & G & \xrightarrow{\pi_Q} & Q \longrightarrow 1 \\
& & \downarrow \chi_N & & \downarrow \chi_G & & \downarrow \sigma_Q \\
e_{(A,C)}: 1 & \longrightarrow & N^{\text{U}(C)}(A)/\text{U}(A) & \longrightarrow & \text{Out}(C, A) & \longrightarrow & \text{Out}^A(C) \longrightarrow 1
\end{array} \tag{8.10}$$

is commutative.

- The composite

$$G \xrightarrow{\chi_G} \text{Out}(C, A) \xrightarrow{\text{res}} \text{Aut}^S(T) \tag{8.11}$$

coincides with  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ .

*Remark 8.3.* In the special case where  $Q$  is the trivial group, this notion of strong normal Deuring embedding comes down to the notion of strong Deuring embedding introduced in Subsection 3.9 above.

Given a strong  $Q$ -normal Deuring embedding  $(A \subseteq C, \chi_G)$  with respect to the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$  and to the group extension (8.1), the composite of  $\chi_G$  with the restriction map  $\text{res}: \text{Out}(C, A) \rightarrow \text{Out}(A)$  yields a  $G$ -normal structure

$$\sigma_G: G \longrightarrow \text{Out}(A) \tag{8.12}$$

on  $A$  relative to the action  $\kappa_G: G \rightarrow \text{Aut}^S(T)$  of  $G$  on  $T$ ; we will refer to this structure as being *associated to the strong  $Q$ -normal Deuring embedding*.

## 8.2 Discussion of the notion of normal Deuring embedding

Recall that  $G$  denotes the fiber product group  $\text{Aut}^S(T) \times_{\text{Aut}(S)} Q$  relative to the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$ , that  $\kappa_G: G \rightarrow \text{Aut}^S(T)$  is the associated obvious homomorphism, and that  $\kappa_G$ , restricted to  $N$ , boils down to the identity  $N \rightarrow \text{Aut}(T|S)$ , cf. (6.1) above.

Let  $A$  be a central  $T$ -algebra, consider an embedding of  $A$  into a central  $S$ -algebra  $C$ , and let  $\sigma_Q: Q \rightarrow \text{Out}(C)$  be a  $Q$ -normal structure on  $C$ . Consider the fiber product group  $B^{A, \sigma_Q} = \text{Aut}^A(C) \times_{\text{Out}(C)} Q$  relative to the  $Q$ -normal structure  $\sigma_Q$  on  $C$ . The following is immediate.

**Proposition 8.4.** *Abstract nonsense identifies the kernel of the canonical homomorphism  $B^{A, \sigma_Q} \rightarrow Q$  with the normal subgroup  $\text{IAut}^A(C)$  of  $\text{Aut}^A(C)$  that consists of the inner automorphisms of  $C$  that map  $A$  to itself. Consequently the data determine a crossed module  $(N^{\text{U}(C)}(A), B^{A, \sigma_Q}, \partial^{A, \sigma_Q})$ , the requisite action of  $B^{A, \sigma_Q}$  on  $N^{\text{U}(C)}(A)$  being induced from the canonical homomorphism  $B^{A, \sigma_Q} \rightarrow \text{Aut}^A(C)$ , in such a way that the sequence*

$$0 \longrightarrow \text{U}(S) \longrightarrow N^{\text{U}(C)}(A) \xrightarrow{\partial^{A, \sigma_Q}} B^{A, \sigma_Q} \longrightarrow Q \tag{8.13}$$

is exact. □

Since  $G = \text{Aut}^S(T) \times_{\text{Aut}(S)} Q$  (relative to the action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$ ), and since the composite  $Q \xrightarrow{\sigma_Q} \text{Out}(C) \xrightarrow{\text{res}} \text{Aut}(S)$  coincides with  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ , by abstract nonsense, the combined homomorphism

$$B^{A,\sigma_Q} \xrightarrow{\text{can}} \text{Aut}^A(C) \xrightarrow{\text{res}} \text{Aut}^S(T)$$

and the canonical homomorphism  $\text{can}: B^{A,\sigma_Q} \rightarrow Q$  induce a homomorphism

$$\pi_G: B^{A,\sigma_Q} = \text{Aut}^A(C) \times_{\text{Out}(C)} Q \longrightarrow \text{Aut}^S(T) \times_{\text{Aut}(S)} Q = G. \quad (8.14)$$

The following is again immediate.

**Proposition 8.5.** *The embedding of  $A$  into  $C$  is a  $Q$ -normal Deuring embedding with respect to the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$  on  $C$  and to the group extension (8.1) if and only if the homomorphism  $\pi_G: B^{A,\sigma_Q} \rightarrow G$  is surjective.  $\square$*

Whether or not the homomorphism  $\pi_G$  is surjective, we shall now determine the kernel of  $\pi_G$ . To this end, let  $\text{Aut}^A(C|T)$  denote the subgroup of  $\text{Aut}^A(C)$  that consists of the automorphisms in  $\text{Aut}^A(C)$  that are the identity on  $T$ . Since  $T$  coincides with the center of  $A$ , restriction induces a homomorphism from  $\text{Aut}^A(C)$  to  $\text{Aut}(T)$ , and since  $S$  coincides with the center of  $C$ , the values of this restriction map lie in the subgroup  $\text{Aut}^S(T)$  of  $\text{Aut}(T)$  that consists of the automorphisms of  $T$  which map  $S$  to itself. Thus, all told, restriction yields an exact sequence

$$1 \longrightarrow \text{Aut}^A(C|T) \longrightarrow \text{Aut}^A(C) \xrightarrow{\text{res}} \text{Aut}^S(T) \quad (8.15)$$

of groups.

Consider the fiber product groups

$$B^{A,\kappa_G} = \text{Aut}^A(C) \times_{\text{Aut}^S(T)} G, \quad B^{A,\kappa_Q} = \text{Aut}^A(C) \times_{\text{Aut}(S)} Q,$$

relative to the homomorphisms  $\kappa_G: G \rightarrow \text{Aut}^S(T)$  and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ , respectively, and let  $\text{can}: B^{A,\kappa_G} \rightarrow G$  denote the canonical homomorphism. Since  $G$  is the fiber product group  $\text{Aut}^S(T) \times_{\text{Aut}(S)} Q$  with respect to the homomorphism  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ , by abstract nonsense, the canonical homomorphism from  $B^{A,\kappa_G}$  to  $B^{A,\kappa_Q}$  is an isomorphism. Moreover, the exact sequence (8.15) induces an exact sequence

$$1 \longrightarrow \text{Aut}^A(C|T) \longrightarrow B^{A,\kappa_G} \xrightarrow{\text{can}} G \quad (8.16)$$

of groups in such a way that

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Aut}^A(C|T) & \longrightarrow & B^{A,\kappa_G} & \xrightarrow{\text{can}} & G \\ & & \parallel & & \downarrow & & \downarrow \kappa_G \\ 1 & \longrightarrow & \text{Aut}^A(C|T) & \longrightarrow & \text{Aut}^A(C) & \xrightarrow{\text{res}} & \text{Aut}^S(T) \end{array}$$

is a commutative diagram with exact rows.

Abstract nonsense yields a canonical homomorphism

$$\text{Aut}^A(C) \times_{\text{Out}(C)} Q = B^{A,\sigma_Q} \longrightarrow B^{A,\kappa_Q} = \text{Aut}^A(C) \times_{\text{Aut}(S)} Q$$

and hence a canonical homomorphism  $B^{A,\sigma_Q} \rightarrow B^{A,\kappa_G}$  whose composite  $B^{A,\sigma_Q} \rightarrow B^{A,\kappa_G} \xrightarrow{\text{can}} G$  with  $\text{can}: B^{A,\kappa_G} \rightarrow G$  coincides with  $\pi_G: B^{A,\sigma_Q} \rightarrow G$ .

**Proposition 8.6.** (i) The homomorphism  $B^{A,\sigma_Q} \rightarrow B^{A,\kappa_G}$  is injective.

(ii) Under the identification of  $B^{A,\sigma_Q}$  with its isomorphic image in the group  $B^{A,\kappa_G}$ , the group  $\text{Aut}^A(C|T)$  being identified with its isomorphic image in  $B^{A,\kappa_G}$  via (8.16), the kernel of  $\pi_G: B^{A,\sigma_Q} \rightarrow G$  gets identified with the normal subgroup of  $\text{Aut}^A(C|T)$  that consists of the automorphisms in  $\text{Aut}^A(C|T)$  that are inner automorphisms of  $C$ .

(iii) Consequently the canonical homomorphism from the centralizer  $C^{\text{U}(C)}(T)$  of  $T$  in  $\text{U}(C)$  to  $\text{Aut}^A(C|T)$  yields a surjective homomorphism

$$C^{\text{U}(C)}(T) \longrightarrow \ker(\pi_G: B^{A,\sigma_Q} \rightarrow G).$$

*Proof.* Since the canonical homomorphism  $B^{A,\kappa_G} \rightarrow B^{A,\kappa_Q}$  is an isomorphism, the right-hand square in the commutative diagram

$$\begin{array}{ccccc} B^{A,\sigma_Q} & \longrightarrow & B^{A,\kappa_G} & \xrightarrow{\text{can}} & Q \\ \downarrow & & \downarrow & & \downarrow \kappa_Q \\ \text{Aut}^A(C) & \xlongequal{\quad} & \text{Aut}^A(C) & \xrightarrow{\text{res}} & \text{Aut}(S) \end{array}$$

is a pull back square, and hence inspection of the diagram reveals that the homomorphism  $B^{A,\sigma_Q} \rightarrow B^{A,\kappa_G}$  is injective. This establishes (i).

To justify (ii), we note first that the kernel of  $\text{Aut}^A(C) \rightarrow \text{Out}(C)$  is the normal subgroup  $\text{IAut}^A(C)$  of  $\text{Aut}^A(C)$  that consists of the inner automorphisms of  $C$  that map  $A$  to itself. Since the group  $B^{A,\sigma_Q}$  is the fiber product group  $B^{A,\sigma_Q} = \text{Aut}^A(C) \times_{\text{Out}(C)} Q$ , abstract nonsense identifies the kernel of the canonical homomorphism  $B^{A,\sigma_Q} \rightarrow Q$  with  $\text{IAut}^A(C)$ , and it is immediate that  $\ker(\pi_G)$  is a subgroup of  $\text{IAut}^A(C) = \ker(B^{A,\sigma_Q} \rightarrow Q)$ . On the other hand,  $B^{A,\sigma_Q}$  being identified with the corresponding subgroup of  $B^{A,\kappa_G}$ , the kernel of  $\pi_G: B^{A,\sigma_Q} \rightarrow G$  gets identified with the intersection  $B^{A,\sigma_Q} \cap \text{Aut}^A(C|T) \subseteq B^{A,\kappa_G}$  and hence with the intersection

$$\text{IAut}^A(C) \cap \text{Aut}^A(C|T) \subseteq B^{A,\kappa_G}.$$

Consequently the kernel of  $\pi_G$  gets identified with the normal subgroup of  $\text{Aut}^A(C|T)$  that consists of the automorphisms in  $\text{Aut}^A(C|T)$  that are inner automorphisms of  $C$ .

Finally, statement (iii) is an immediate consequence of (ii).  $\square$

**Proposition 8.7.** Suppose that the embedding of  $A$  into  $C$  is a  $Q$ -normal Deuring embedding with respect to the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$  on  $C$  and to the group extension (8.1).

(i) The surjective homomorphism (8.14) yields a crossed 2-fold extension

$$e_{(C,\sigma_Q)}^{A,T}: 0 \longrightarrow \text{U}(S) \longrightarrow C^{\text{U}(C)}(T) \xrightarrow{\partial^{A,T,\sigma_Q}} B^{A,\sigma_Q} \xrightarrow{\pi_G} G \longrightarrow 1. \quad (8.17)$$

(ii) The values of the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$  on  $C$  lie in the subgroup  $\text{Out}^A(C)$  ( $= \text{coker}(\partial_C^N: N^{\text{U}(C)}(A) \longrightarrow \text{Aut}^A(C))$ , cf. (8.8)).

*Proof.* Statement (i) is an immediate consequence of Proposition 8.5 and Proposition 8.6 (iii). Moreover, the diagram

$$\begin{array}{ccccc} N^{\mathcal{U}(C)}(A) & \xrightarrow{\partial^{A,\sigma_Q}} & B^{A,\sigma_Q} & \longrightarrow & Q \\ \parallel & & \downarrow & & \downarrow \sigma_Q \\ N^{\mathcal{U}(C)}(A) & \xrightarrow{\partial_C^N} & \text{Aut}^A(C) & \xrightarrow{\text{can}} & \text{Out}(C) \end{array}$$

is commutative and, in view of Proposition 8.5, the canonical homomorphism  $B^{A,\sigma_Q} \rightarrow Q$  is surjective. Consequently the values of  $\sigma_Q: Q \rightarrow \text{Out}(C)$  on  $C$  lie in the subgroup  $\text{Out}^A(C)$  ( $= \text{coker}(\partial_C^N: N^{\mathcal{U}(C)}(A) \rightarrow \text{Aut}^A(C))$ ).  $\square$

Given a  $Q$ -normal Deuring embedding of  $A$  into  $C$  with respect to the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$  on  $C$  and to the group extension (8.1), in view of Proposition 8.7(ii), let

$$e_{(C,\sigma_Q)}^A: 0 \longrightarrow \mathcal{U}(S) \longrightarrow N^{\mathcal{U}(C)}(A) \xrightarrow{\partial^{A,\sigma_Q}} B^{A,\sigma_Q} \longrightarrow Q \longrightarrow 1 \quad (8.18)$$

denote the associated crossed 2-fold extension induced from (8.8) via the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}^A(C)$  on  $C$ ; the underlying sequence of groups and homomorphisms plainly coincides with (8.13). Recall that the Teichmüller complex  $e_{(C,\sigma_Q)}$  of the kind (3.8) associated to the  $Q$ -normal  $S$ -algebra  $(C, \sigma_Q)$  is the crossed 2-fold extension

$$e_{(C,\sigma_Q)}: 0 \longrightarrow \mathcal{U}(S) \longrightarrow \mathcal{U}(C) \xrightarrow{\partial^{\sigma_Q}} B^{\sigma_Q} \longrightarrow Q \longrightarrow 1 \quad (8.19)$$

induced from (8.2) via the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$  on  $C$ . The following is again immediate.

**Proposition 8.8.** *Suppose that the embedding of  $A$  into  $C$  is a  $Q$ -normal Deuring embedding with respect to the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$  on  $C$  and to the group extension (8.1).*

(i) *The inclusion maps  $N^{\mathcal{U}(C)}(A) \rightarrow \mathcal{U}(C)$  and  $B^{A,\sigma_Q} \rightarrow B^{\sigma_Q}$  yield a congruence*

$$\begin{array}{ccccccc} e_{(C,\sigma_Q)}^A: 0 & \longrightarrow & \mathcal{U}(S) & \longrightarrow & N^{\mathcal{U}(C)}(A) & \xrightarrow{\partial^{A,\sigma_Q}} & B^{A,\sigma_Q} \longrightarrow Q \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ e_{(C,\sigma_Q)}: 0 & \longrightarrow & \mathcal{U}(S) & \longrightarrow & \mathcal{U}(C) & \xrightarrow{\partial^{\sigma_Q}} & B^{\sigma_Q} \longrightarrow Q \longrightarrow 1 \end{array} \quad (8.20)$$

*of crossed 2-fold extensions from the crossed 2-fold extension (8.18) to the crossed 2-fold extension (8.19).*

(ii) *The injection  $C^{\mathcal{U}(C)}(T) \rightarrow N^{\mathcal{U}(C)}(A)$  yields the morphism*

$$\begin{array}{ccccccc} e_{(C,\sigma_Q)}^{A,T}: 0 & \longrightarrow & \mathcal{U}(S) & \longrightarrow & C^{\mathcal{U}(C)}(T) & \xrightarrow{\partial^{A,T,\sigma_Q}} & B^{A,\sigma_Q} \longrightarrow G \longrightarrow 1 \\ & & \parallel & & \downarrow & & \parallel \\ e_{(C,\sigma_Q)}^A: 0 & \longrightarrow & \mathcal{U}(S) & \longrightarrow & N^{\mathcal{U}(C)}(A) & \xrightarrow{\partial^{A,\sigma_Q}} & B^{A,\sigma_Q} \longrightarrow Q \longrightarrow 1 \end{array} \quad (8.21)$$

*of crossed 2-fold extensions from the crossed 2-fold extension (8.17) to the crossed 2-fold extension (8.18).*

### 8.3 Results related with the two notions of normal Deuring embedding

**Theorem 8.9.** *Let  $A$  be a central  $T$ -algebra,  $C$  a central  $S$ -algebra, and  $A \subseteq C$  an embedding of  $A$  into  $C$  having the property that  $A$  coincides with the centralizer of  $T$  in  $C$ . Furthermore, let  $\sigma_Q: Q \rightarrow \text{Out}(C)$  be a  $Q$ -normal structure on  $C$ , and suppose that the embedding of  $A$  into  $C$  is a  $Q$ -normal Deuring embedding with respect to  $\sigma_Q$  and to the group extension (8.1). Then the data determine a unique homomorphism  $\chi_G: G \rightarrow \text{Out}(C, A)$  that turns the given  $Q$ -normal Deuring embedding of  $A$  into  $C$  into a strong  $Q$ -normal Deuring embedding of  $A$  into  $C$  with respect to the given data in such a way that, relative to the associated  $G$ -normal structure*

$$\sigma_G: G \xrightarrow{\chi_G} \text{Out}(C, A) \xrightarrow{\text{res}} \text{Out}(A) \quad (8.22)$$

on  $A$ , cf. (8.12),

$$[e_{(A, \sigma_G)}] = \inf[e_{(C, \sigma_Q)}] \in H^3(G, U(T)).$$

*Proof.* Recall that the Teichmüller complex  $e_{(C, \sigma_Q)}$  of the  $Q$ -normal  $S$ -algebra  $(C, \sigma_Q)$ , spelled out above as (8.19), represents the Teichmüller class  $[e_{(C, \sigma_Q)}] \in H^3(Q, U(S))$  of the  $Q$ -normal  $S$ -algebra  $(C, \sigma_Q)$ .

Suppose that the embedding of  $A$  into  $C$  is a  $Q$ -normal Deuring embedding with respect to the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(C)$  on  $C$  and to the group extension (8.1). By Proposition 8.8(i), the crossed 2-fold extension  $e_{(C, \sigma_Q)}^A$ , cf. (8.18), is available and is congruent to  $e_{(C, \sigma_Q)}$ , whence

$$[e_{(C, \sigma_Q)}] = [e_{(C, \sigma_Q)}^A] \in H^3(Q, U(S)).$$

Moreover, by Proposition 8.8(ii), the crossed 2-fold extension (8.17) is available and, since the centralizer of  $A$  in  $C$  coincides with  $T$ , the inclusion  $U(A) \subseteq C^{U(C)}(T)$  identifies the group  $U(A)$  of invertible elements of  $A$  with the centralizer  $C^{U(C)}(T)$  of  $T$  in  $U(C)$ . Hence the crossed 2-fold extension (8.17) has the form

$$e_{(C, \sigma_Q)}^{A, T}: 0 \longrightarrow U(S) \longrightarrow U(A) \xrightarrow{\partial^{A, T, \sigma_Q}} B^{A, \sigma_Q} \longrightarrow G \longrightarrow 1, \quad (8.23)$$

and the injection  $\iota: U(A) \rightarrow N^{U(C)}(A)$  induces the morphism (8.21) of crossed 2-fold extensions in Proposition 8.8(ii); this is a morphism of crossed 2-fold extensions of the kind  $(1, \iota, 1, \pi_Q): e_{(C, \sigma_Q)}^{A, T} \rightarrow e_{(C, \sigma_Q)}^A$ .

The canonical homomorphism  $B^{A, \sigma_Q} = \text{Aut}^A(C) \times_{\text{Out}(C)} Q \rightarrow \text{Aut}^A(C)$  induces a morphism

$$(\text{Id}, \cdot): (U(A), B^{A, \sigma_Q}, \partial^{A, T, \sigma_Q}) \longrightarrow (U(A), \text{Aut}^A(C), \partial_C^A)$$

of crossed modules and hence a homomorphism  $\chi_G: G \rightarrow \text{Out}(C, A)$  such that, with the notation  $i: U(S) \rightarrow U(T)$  for the inclusion,

$$\begin{array}{ccccccccc} e_{(C, \sigma_Q)}^{A, T}: 0 & \longrightarrow & U(S) & \longrightarrow & U(A) & \xrightarrow{\partial^{A, T, \sigma_Q}} & B^{A, \sigma_Q} & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow i & & \parallel & & \downarrow & & \downarrow \chi_G & & \\ e_A: 0 & \longrightarrow & U(T) & \longrightarrow & U(A) & \xrightarrow{\partial_C^A} & \text{Aut}^A(C) & \longrightarrow & \text{Out}(C, A) & \longrightarrow & 1 \end{array}$$

is a morphism of crossed 2-fold extensions from (8.23) to (3.1). The homomorphism  $\chi_G$  turns the given  $Q$ -normal Deuring embedding of  $C$  into  $A$  into a strong  $Q$ -normal Deuring embedding of  $C$  into  $A$  with respect to the given data.

The  $G$ -normal structure  $\sigma_G: G \xrightarrow{\chi_G} \text{Out}(C, A) \xrightarrow{\text{res}} \text{Out}(A)$  associated to the strong  $Q$ -normal Deuring embedding, in turn, induces a morphism

$$\begin{array}{ccccccccc} e_{(C, \sigma_Q)}^{A, T}: 0 & \longrightarrow & U(S) & \longrightarrow & U(A) & \xrightarrow{\partial^{A, T, \sigma_Q}} & B^{A, \sigma_Q} & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow i & & \parallel & & \downarrow & & \parallel \\ e_{(A, \sigma_G)}: 0 & \longrightarrow & U(T) & \longrightarrow & U(A) & \xrightarrow{\partial^{\sigma_G}} & B^{\sigma_G} & \longrightarrow & G \longrightarrow 1 \end{array}$$

of crossed 2-fold extensions from (8.23) to the corresponding crossed 2-fold extension  $e_{(A, \sigma_G)}$  of the kind (3.8). Consequently  $[e_{(A, \sigma_G)}] = \inf[e_{(C, \sigma_Q)}]$ .  $\square$

Theorem 8.9 has a converse; this converse sort of a characterizes the Teichmüller classes in  $H^3(Q, U(S))$ .

**Theorem 8.10.** *Let  $k \in H^3(Q, U(S))$ , let  $A$  be a central  $T$ -algebra, and let  $\sigma_G: G \rightarrow \text{Out}(A)$  be a  $G$ -normal structure on  $A$  relative to the action  $\kappa_G: G \rightarrow \text{Aut}^S(T)$  of  $G$  on  $T$ . Suppose that*

$$\inf(k) = [e_{(A, \sigma_G)}] \in H^3(G, U(T)).$$

*Then there is a  $Q$ -normal  $S$ -central crossed product algebra*

$$(C, \sigma_Q) = ((A, N, e, \vartheta), \sigma_Q)$$

*related with the other data in the following way.*

- *The  $Q$ -normal algebra  $(C, \sigma_Q) = ((A, N, e, \vartheta), \sigma_Q)$  has Teichmüller class  $k$ ;*
- *once the  $Q$ -normal algebra  $((A, N, e, \vartheta), \sigma_Q)$  has been fixed, the data determine a homomorphism  $\chi_G: G \rightarrow \text{Out}(C, A)$  that turns the obvious embedding of  $A$  into  $(A, N, e, \vartheta)$  into a strong  $Q$ -normal Deuring embedding with respect to  $\sigma_Q: Q \rightarrow \text{Out}(A, N, e, \vartheta)$  and to the group extension (8.1);*
- *the associated  $G$ -normal structure*

$$G \xrightarrow{\chi_G} \text{Out}(C, A) \xrightarrow{\text{res}} \text{Out}(A) \tag{8.24}$$

*on  $A$ , cf. (8.12), and the given  $G$ -normal structure  $\sigma_G: G \rightarrow \text{Out}(A)$  on  $A$  coincide.*

**Complement 8.11.** *In the situation of Theorem 8.10, if  $A$  is an Azumaya  $T$ -algebra, the algebra  $(A, N, e, \vartheta)$  is an Azumaya  $S$ -algebra.*

*Remark 8.12.* In the special case where  $\inf(k) = 0$ , the argument to be given comes down to that given for the statement of Theorem 7.5, and this theorem is in fact a special case of Theorem 8.10.

*Proof.* For convenience, we split the reasoning into the series of Propositions 8.13 - 8.15 below.



Consider a  $G$ -normal central  $T$ -algebra  $(A, \sigma_G)$ , and denote by  $\sigma_N: N \rightarrow \text{Out}(A)$  the restriction of  $\sigma_G: G \rightarrow \text{Out}(A)$  to  $N$  so that  $(A, \sigma_N)$  is an  $N$ -normal central  $T$ -algebra. The obvious unlabeled vertical arrow and the injection  $i^N$  turn

$$\begin{array}{ccccccccc} e_{(A, \sigma_N)}: 0 & \longrightarrow & U(T) & \longrightarrow & U(A) & \xrightarrow{\partial^{\sigma_N}} & B^{\sigma_N} & \longrightarrow & N & \longrightarrow & 1 \\ & & \parallel & & \parallel & & \downarrow & & i^N \downarrow & & \\ e_{(A, \sigma_G)}: 0 & \longrightarrow & U(T) & \longrightarrow & U(A) & \xrightarrow{\partial^{\sigma_G}} & B^{\sigma_G} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

into a commutative diagram having as its rows the (exact) Teichmüller complexes  $e_{(A, \sigma_N)}$  and  $e_{(A, \sigma_G)}$  of  $(A, \sigma_N)$  and  $(A, \sigma_G)$ , respectively. Consequently the combined homomorphism

$$B^{\sigma_G} \longrightarrow G \xrightarrow{\pi_Q} Q$$

yields a group extension

$$1 \longrightarrow B^{\sigma_N} \longrightarrow B^{\sigma_G} \longrightarrow Q \longrightarrow 1. \quad (8.25)$$

Let

$$\hat{e}: 0 \longrightarrow U(T) \longrightarrow U(A) \longrightarrow U(A)/U(T) \longrightarrow 1 \quad (8.26)$$

and

$$1 \longrightarrow U(A)/U(T) \xrightarrow{\phi} B^{\sigma_N} \longrightarrow N \longrightarrow 1$$

be the obvious group extensions so that splicing them yields the Teichmüller complex

$$e_{(A, \sigma_N)}: 0 \longrightarrow U(T) \longrightarrow U(A) \longrightarrow B^{\sigma_N} \longrightarrow N \longrightarrow 1$$

of  $(A, \sigma_N)$ . We denote the resulting morphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(A)/U(T) & \longrightarrow & B^{\sigma_G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \phi & & \parallel & & \downarrow \pi_Q & & \\ 1 & \longrightarrow & B^{\sigma_N} & \longrightarrow & B^{\sigma_G} & \longrightarrow & Q & \longrightarrow & 1 \end{array} \quad (8.27)$$

of group extensions by  $\Phi$ .

Consider the Teichmüller complex

$$e_{(A, \sigma_G)}: 0 \longrightarrow U(T) \longrightarrow U(A) \xrightarrow{\partial^{\sigma_G}} B^{\sigma_G} \longrightarrow G \longrightarrow 1,$$

associated to the given  $G$ -normal structure  $\sigma_G: G \rightarrow \text{Out}(A)$  on  $A$ , cf. (3.8). Since  $U(T)$  is a central subgroup of  $U(A)$ , the group extension  $\hat{e}$  spelled out above as (8.26) is a central extension and, as noted in Proposition 7.2,  $G$ -crossed pair structures on  $\hat{e}$  are equivalent to  $B^{\sigma_G}$ -actions on  $U(A)$  that turn  $U(A) \rightarrow B^{\sigma_G}$  into a crossed module. Thus the action of  $B^{\sigma_G}$  on  $U(A)$  that results from the given  $G$ -normal structure  $\sigma_G: G \rightarrow \text{Out}(A)$  via the associated crossed 2-fold extension  $e_{(A, \sigma_G)}$  induces a crossed pair structure  $\hat{\psi}: G \rightarrow \text{Out}_{B^{\sigma_G}}(\hat{e})$  on  $\hat{e}$  with respect to the group extension

$$1 \longrightarrow U(A)/U(T) \longrightarrow B^{\sigma_G} \longrightarrow G \longrightarrow 1$$

and the  $G$ -module  $U(T)$ . Then the canonical homomorphism  $\hat{\gamma}: \text{Aut}_{B^{\sigma_G}}(\hat{e}) \rightarrow \text{Aut}(A)$  yields a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(T) & \longrightarrow & U(A) & \longrightarrow & \text{Aut}_{B^{\sigma_G}}(\hat{e}) & \longrightarrow & \text{Out}_{B^{\sigma_G}}(\hat{e}) & \longrightarrow & 1 \\ & & \parallel & & \parallel & & \downarrow \hat{\gamma} & & \downarrow \hat{\gamma}_{\#} & & \\ 0 & \longrightarrow & U(T) & \longrightarrow & U(A) & \longrightarrow & \text{Aut}(A) & \longrightarrow & \text{Out}(A) & \longrightarrow & 1 \end{array}$$

of crossed 2-fold extensions such that the composite

$$G \xrightarrow{\hat{\psi}} \text{Out}_{B^{\sigma_G}}(\hat{e}) \xrightarrow{\hat{\gamma}_{\#}} \text{Out}(A) \quad (8.28)$$

coincides with  $\sigma_G: G \rightarrow \text{Out}(A)$ .

**Proposition 8.13.** *Let  $k \in H^3(Q, U(S))$ , let  $(A, \sigma_G)$  be a  $G$ -normal central  $T$ -algebra, and suppose that  $\inf(k) = [e_{(A, \sigma_G)}] \in H^3(G, U(T))$ . Then there is a group extension*

$$\tilde{e}: 0 \longrightarrow U(T) \longrightarrow \Gamma \longrightarrow B^{\sigma_N} \longrightarrow 1$$

together with a crossed pair structure  $\tilde{\psi}: Q \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e})$  on  $\tilde{e}$  with respect to the group extension (8.25) and to the  $B^{\sigma_G}$ -module  $U(T)$ , the requisite module structure being induced by the map  $B^{\sigma_G} \rightarrow G$  in  $e_{(A, \sigma_G)}$ , related with the other data in the following way, where  $B^{\tilde{\psi}}$  denotes the fiber product group  $\text{Aut}_{B^{\sigma_G}}(\tilde{e}) \times_{\text{Out}_{B^{\sigma_G}}(\tilde{e})} Q$  with respect to  $\tilde{\psi}: Q \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e})$ .

(i) The crossed 2-fold extension

$$e_{\tilde{\psi}}: 0 \longrightarrow U(S) \longrightarrow \Gamma \longrightarrow B^{\tilde{\psi}} \longrightarrow Q \longrightarrow 1$$

associated to the crossed pair  $(\tilde{e}, \tilde{\psi})$ , cf. (7.5), represents  $k$ .

(ii) Relative to the obvious actions of the group  $\text{Aut}_{B^{\sigma_G}}(\tilde{e}) (\subseteq \text{Aut}(\Gamma) \times B^{\sigma_G})$  on the groups  $U(T), U(A), U(A)/U(T), \Gamma, B^{\sigma_N}$  and  $N$ , the extension group  $\Gamma$  in  $\tilde{e}$  fits into a commutative diagram of  $\text{Aut}_{B^{\sigma_G}}(\tilde{e})$ -groups with exact rows and columns as follows:

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ \hat{e}: 0 & \longrightarrow & U(T) & \longrightarrow & U(A) & \longrightarrow & U(A)/U(T) \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \phi \\ \tilde{e}: 0 & \longrightarrow & U(T) & \longrightarrow & \Gamma & \longrightarrow & B^{\sigma_N} \longrightarrow 1 \\ & & & & \downarrow & & \downarrow \\ & & & & N & \xlongequal{\quad} & N \\ & & & & \downarrow & & \downarrow \\ & & & & 1 & & 1 \end{array} \quad (8.29)$$

*Proof.* By [Hue81b, Theorem 2], the morphism 8.27 of group extensions induces a morphism for the corresponding eight term exact sequences in group cohomology constructed

in [Hue81b]. In particular,  $\Phi$  induces the commutative diagram

$$\begin{array}{ccccccc} H^2(B^{\sigma_G}, U(T)) & \longrightarrow & \text{Xpext}(B^{\sigma_G}, B^{\sigma_N}; U(T)) & \xrightarrow{\Delta} & H^3(Q, U(S)) & \longrightarrow & H^3(B^{\sigma_G}, U(T)) \\ \parallel & & \downarrow \Phi^* & & \downarrow \text{inf} & & \parallel \\ H^2(B^{\sigma_G}, U(T)) & \longrightarrow & \text{Xpext}(B^{\sigma_G}, U(A)/U(T); U(T)) & \xrightarrow{\Delta} & H^3(G, U(T)) & \longrightarrow & H^3(B^{\sigma_G}, U(T)). \end{array}$$

By the construction of  $\Delta$ , cf. Subsection 7.1 above or [Hue81b, Subsection 1.2],  $\Delta[(\hat{e}, \hat{\psi})] = [e_{(A, \sigma_G)}]$ , and so, by exactness,  $\text{inf}(k) = [e_{(A, \sigma_G)}]$  goes to zero in  $H^3(B^{\sigma_G}, U(T))$ . Therefore  $k$  goes to zero in  $H^3(B^{\sigma_G}, U(T))$ , and hence there is a group extension

$$\tilde{e}: 0 \longrightarrow U(T) \longrightarrow \Gamma \longrightarrow B^{\sigma_N} \longrightarrow 1$$

of the asserted kind together with a crossed pair structure  $\tilde{\psi}: Q \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e})$  on  $\tilde{e}$  with respect to the group extension (8.25) and to the  $B^{\sigma_G}$ -module  $U(T)$  whose  $B^{\sigma_G}$ -module structure is induced by the projection  $B^{\sigma_G} \rightarrow G$  in  $e_{(A, \sigma_G)}$  so that

$$\Delta[(\tilde{e}, \tilde{\psi})] = k \in H^3(Q, U(S));$$

moreover, making a suitable choice of  $(\tilde{e}, \tilde{\psi})$  by means of some diagram chase if need be, we can arrange for  $[(\tilde{e}, \tilde{\psi})]$  to go to  $[(\hat{e}, \hat{\psi})]$  in the sense that

$$\Phi^*[(\tilde{e}, \tilde{\psi})] = [(\hat{e}, \hat{\psi})] \in \text{Xpext}(B^{\sigma_G}, U(A)/U(T); U(T)).$$

The crossed pair  $(\tilde{e}, \tilde{\psi})$  has the asserted properties. For  $\Delta[(\tilde{e}, \tilde{\psi})] = [e_{\tilde{\psi}}]$  by definition, and so assertion (i) holds. Moreover, since  $\Phi^*[(\tilde{e}, \tilde{\psi})] = [(\hat{e}, \hat{\psi})]$ , assertion (ii) holds as well. The details are as follows, cf. [Hue81b, Subsection 2.2].

Since  $\Phi^*[(\tilde{e}, \tilde{\psi})] = [(\hat{e}, \hat{\psi})]$ , we may identify  $(\hat{e}, \hat{\psi})$  with the induced crossed pair  $(\tilde{e}\Phi, \tilde{\psi}^\Phi)$ , cf. [Hue81b]. Recall that  $\tilde{e}\Phi$  is the group extension induced from  $\tilde{e}$  via the injective homomorphism  $\phi: U(A)/U(T) \rightarrow B^{\sigma_N}$ ; since  $\phi$  identifies  $U(A)/U(T)$  with the kernel of  $B^{\sigma_N} \rightarrow N$ , with the notation  $U = \ker(\Gamma \rightarrow N)$ , the induced group extension  $\tilde{e}\Phi$  may be written as

$$\tilde{e}\Phi: 0 \longrightarrow U(T) \longrightarrow U \longrightarrow U(A)/U(T) \longrightarrow 1.$$

To explain the induced crossed pair structure  $\tilde{\psi}^\Phi: G \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e}\Phi)$ , we note first that the injection  $U \rightarrow \Gamma$  induces a morphism

$$\begin{array}{ccccccc} 0 \longrightarrow & U(S) & \longrightarrow & U & \longrightarrow & \text{Aut}_{B^{\sigma_G}}(\tilde{e}) & \longrightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G \longrightarrow 1 \\ & \parallel & & \downarrow & & \parallel & \downarrow \\ 0 \longrightarrow & U(S) & \longrightarrow & \Gamma & \longrightarrow & \text{Aut}_{B^{\sigma_G}}(\tilde{e}) & \longrightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e}) \longrightarrow 1 \end{array}$$

of crossed 2-fold extensions. Moreover, restriction of the operators on  $\Gamma$  to  $U$  yields a homomorphism

$$\text{res}: \text{Aut}_{B^{\sigma_G}}(\tilde{e}) \longrightarrow \text{Aut}_{B^{\sigma_G}}(\tilde{e}\Phi),$$

and this homomorphism, in turn, yields a morphism

$$\begin{array}{ccccccc} 0 \longrightarrow & U(S) & \longrightarrow & U & \longrightarrow & \text{Aut}_{B^{\sigma_G}}(\tilde{e}) & \longrightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G \longrightarrow 1 \\ & \downarrow & & \parallel & & \downarrow \text{res} & \downarrow \text{res}_\flat \\ 0 \longrightarrow & U(T) & \longrightarrow & U & \longrightarrow & \text{Aut}_{B^{\sigma_G}}(\tilde{e}\Phi) & \longrightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e}\Phi) \longrightarrow 1 \end{array}$$

of crossed 2-fold extensions. The crossed pair structure  $\tilde{\psi}^\Phi: G \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e}\Phi)$  is the composite

$$G \xrightarrow{\tilde{\psi}_G} \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G \xrightarrow{\text{res}_\flat} \text{Out}_{B^{\sigma_G}}(\tilde{e}\Phi)$$

of  $\text{res}_\flat$  with the canonical lift of the crossed pair structure  $\tilde{\psi}: Q \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e})$  on  $\tilde{e}$  to a homomorphism  $\tilde{\psi}_G: G \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G$ ; see [Hue81b, Propositions 2.3 and 2.4]. The identity  $\Phi^*[(\tilde{e}, \tilde{\psi})] = [(\hat{e}, \hat{\psi})]$  means that the two crossed pairs  $(\hat{e}, \hat{\psi})$  and  $(\tilde{e}\Phi, \tilde{\psi}^\Phi)$  are congruent as crossed pairs. Thus we may take  $U$  to be  $U(A)$  such that the following hold:

- The injection  $U(A) \rightarrow \Gamma$  induces a morphism  $\hat{e} \rightarrow \tilde{e}$  of group extensions whose restriction to  $U(T)$  is the identity, as displayed in diagram (8.29) above, and
- the crossed pair structure  $\hat{\psi}: G \rightarrow \text{Out}_{B^{\sigma_G}}(\hat{e})$  on  $\hat{e}$  is the composite

$$G \xrightarrow{\tilde{\psi}_G} \text{Out}_{B^{\sigma_G}}(\tilde{e}) \xrightarrow{\text{res}_\sharp} \text{Out}_{B^{\sigma_G}}(\hat{e}) \quad (8.30)$$

of  $\tilde{\psi}_G$  with the homomorphism  $\text{res}_\sharp: \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G \rightarrow \text{Out}_{B^{\sigma_G}}(\hat{e})$  induced by the obvious restriction homomorphism  $\text{res}: \text{Aut}_{B^{\sigma_G}}(\tilde{e}) \rightarrow \text{Aut}_{B^{\sigma_G}}(\hat{e})$ .

The morphism  $\hat{e} \rightarrow \tilde{e}$  of group extensions yields the commutative diagram (8.29) and, by construction, this is a commutative diagram of  $\text{Aut}_{B^{\sigma_G}}(\tilde{e})$ -groups.  $\square$

We continue the proof of Theorem 8.10. Maintaining the hypotheses of Proposition 8.13, we write

$$e: 1 \longrightarrow U(A) \xrightarrow{j} \Gamma \longrightarrow N \longrightarrow 1 \quad (8.31)$$

for the group extension that arises as the middle column of diagram (8.29) and denote by  $\vartheta: \Gamma \rightarrow \text{Aut}(A)$  the combined homomorphism

$$\Gamma \longrightarrow B^{\sigma_N} \longrightarrow \text{Aut}(A).$$

Consider the crossed product algebra  $(A, N, e, \vartheta)$ . By construction

$$(A, N, e, \vartheta) = A^t \Gamma / \langle a - j(a), a \in U(A) \rangle,$$

cf. Section 4 above. By Proposition 4.3(iv), since  $T|S$  is a Galois extension of commutative rings with Galois group  $N$ , the group  $\Gamma$  now gets identified with the normalizer  $N^{U(A, N, e, \vartheta)}(A)$  of  $A$  in the crossed product algebra  $(A, N, e, \vartheta)$ .

Recall the notation  $B^{\sigma_G}$  for the fiber product group  $\text{Aut}(A) \times_{\text{Out}(A)} G$  with respect to the given  $G$ -normal structure  $\sigma_G: G \rightarrow \text{Out}(A)$  on  $A$ , cf. Subsection 3.4 above. Furthermore, recall from Subsection 7.1 above that  $\text{Aut}_{B^{\sigma_G}}(\tilde{e})$  denotes the subgroup of  $\text{Aut}(\Gamma) \times B^{\sigma_G}$  that consists of the pairs  $(\alpha, x)$  which render the diagram

$$\begin{array}{ccccccc} \tilde{e}: 0 & \longrightarrow & U(T) & \longrightarrow & \Gamma & \longrightarrow & B^{\sigma_N} \longrightarrow 1 \\ & & \ell_x \downarrow & & \alpha \downarrow & & i_x \downarrow \\ \tilde{e}: 0 & \longrightarrow & U(T) & \longrightarrow & \Gamma & \longrightarrow & B^{\sigma_N} \longrightarrow 1 \end{array}$$

commutative; here, given  $x \in B^{\sigma_G}$ , the notation  $i_x: B^{\sigma_N} \rightarrow B^{\sigma_N}$  refers to conjugation by  $x \in B^{\sigma_G}$  and  $\ell_x: U(T) \rightarrow U(T)$  to the canonical action of  $B^{\sigma_G}$  on  $U(T)$  (recall that  $T$  denotes the center of  $A$ ) induced from the action of  $B^{\sigma_G}$  on  $A$  and hence on  $U(T)$  via the canonical homomorphism  $B^{\sigma_G} \rightarrow \text{Aut}(A)$ .

**Proposition 8.14.** *The rule*

$$^{(\alpha, x)}(ay) = {}^x a^\alpha y, \quad a \in A, \quad y \in \Gamma, \quad (8.32)$$

where  $(\alpha, x) \in \text{Aut}_{B^{\sigma_G}}(\tilde{e}) (\subseteq \text{Aut}(\Gamma) \times B^{\sigma_G} \subseteq \text{Aut}(\Gamma) \times \text{Aut}(A) \times G)$ , yields a homomorphism

$$\gamma: \text{Aut}_{B^{\sigma_G}}(\tilde{e}) \longrightarrow \text{Aut}^A(A, N, e, \vartheta) \quad (8.33)$$

which, in turn, yields morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(S) & \longrightarrow & \Gamma & \longrightarrow & \text{Aut}_{B^{\sigma_G}}(\tilde{e}) & \longrightarrow & \text{Out}_{B^{\sigma_G}}(\tilde{e}) & \longrightarrow & 1 \\ & & \parallel & & \parallel & & \downarrow \gamma & & \downarrow \gamma_\# & & \\ 0 & \longrightarrow & U(S) & \longrightarrow & N^{U(A, N, e, \vartheta)}(A) & \longrightarrow & \text{Aut}^A(A, N, e, \vartheta) & \longrightarrow & \text{Out}^A(A, N, e, \vartheta) & \longrightarrow & 1 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(S) & \longrightarrow & U(A) & \longrightarrow & \text{Aut}_{B^{\sigma_G}}(\tilde{e}) & \longrightarrow & \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G & \longrightarrow & 1 \\ & & \parallel & & \parallel & & \downarrow \gamma & & \downarrow \gamma_b & & \\ 0 & \longrightarrow & U(S) & \longrightarrow & U(A) & \longrightarrow & \text{Aut}^A(A, N, e, \vartheta) & \longrightarrow & \text{Out}((A, N, e, \vartheta), A) & \longrightarrow & 1 \end{array}$$

of crossed 2-fold extensions. Furthermore, the homomorphisms  $\gamma_\#$ ,  $\gamma_b$ , and the obvious unlabeled homomorphisms render the diagram

$$\begin{array}{ccc} \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G & \longrightarrow & \text{Out}_{B^{\sigma_G}}(\tilde{e}) \\ \gamma_b \downarrow & & \downarrow \gamma_\# \\ \text{Out}((A, N, e, \vartheta), A) & \longrightarrow & \text{Out}^A(A, N, e, \vartheta) \end{array} \quad (8.34)$$

commutative.

*Proof.* The left  $A$ -module that underlies the twisted group ring  $A^t \Gamma$  is the free  $A$ -module having  $\Gamma$  as an  $A$ -basis, whence it is manifest that the rule (8.32) yields an action of the group  $\text{Aut}_{B^{\sigma_G}}(\tilde{e})$  on that left  $A$ -module.

Next we shall show that the  $\text{Aut}_{B^{\sigma_G}}(\tilde{e})$ -action on the left  $A$ -module that underlies the twisted group ring  $A^t \Gamma$  is compatible with the multiplicative structure of  $A^t \Gamma$ . To this end, consider the crossed module  $(\Gamma, \text{Aut}_{B^{\sigma_G}}(\tilde{e}), \beta)$ , cf. the middle columns of the commutative diagram (7.3) above. Since  $\beta: \Gamma \rightarrow \text{Aut}_{B^{\sigma_G}}(\tilde{e})$  is a morphism of  $\text{Aut}_{B^{\sigma_G}}(\tilde{e})$ -groups, given  $y \in \Gamma$  and  $(\alpha, x) \in \text{Aut}_{B^{\sigma_G}}(\tilde{e})$ ,

$$\beta(^{(\alpha, x)}y) = (\alpha, x)\beta(y)(\alpha, x)^{-1} \in \text{Aut}_{B^{\sigma_G}}(\tilde{e}). \quad (8.35)$$

Now it is manifest that the action  $\vartheta: \Gamma \rightarrow \text{Aut}(A)$  of  $\Gamma$  on  $A$  factors through  $\beta$ , that is, with the notation can for the canonical homomorphism,  $\vartheta$  coincides with the combined homomorphism

$$\Gamma \xrightarrow{\beta} \text{Aut}_{B^{\sigma_G}}(\tilde{e}) \xrightarrow{\text{can}} \text{Aut}(A).$$

Hence, given  $(\alpha, x) \in \text{Aut}_{B^{\sigma_G}}(\tilde{e}) (\subseteq \text{Aut}(\Gamma) \times B^{\sigma_G})$ ,  $b \in A$ , and  $y \in \Gamma$ ,

$$x\vartheta(y)x^{-1}b = \vartheta(^{\alpha}y)b. \quad (8.36)$$

Thus, given  $(\alpha, x) \in \text{Aut}_{B^{\sigma_G}}(\tilde{e})$ ,  $y \in \Gamma$ ,  $a \in A$ , in view of (8.36) we conclude

$$^{(\alpha, x)}(ya) = ^{(\alpha, x)}(\vartheta(y)a y) = (x\vartheta(y)x^{-1}a)^\alpha y = (\vartheta(^{\alpha}y)x a)^\alpha y = ^\alpha y^x a.$$

Consequently the rule (8.32) yields an action of  $\text{Aut}_{B^{\sigma_G}}(\tilde{e})$  on the algebra  $A^t\Gamma$ .

Finally, to show that the action of  $\text{Aut}_{B^{\sigma_G}}(\tilde{e})$  on the algebra  $A^t\Gamma$  preserves the two-sided ideal  $\langle a - j(a), a \in U(A) \rangle$  in  $A^t\Gamma$ , let  $a \in U(A)$  and  $(\alpha, x) \in \text{Aut}_{B^{\sigma_G}}(\tilde{e})$ . In view of Proposition 8.13 (ii),  $j(^x a) = ^\alpha(j(a))$ , whence

$$^{(\alpha, x)}(a - j(a)) = (^x a - j(^x a)).$$

□

With respect to the crossed pair structure  $\tilde{\psi}: Q \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e})$  on  $\tilde{e}$ , the fiber product group  $B^{\tilde{\psi}} = \text{Aut}_{B^{\sigma_G}}(\tilde{e}) \times_{\text{Out}_{B^{\sigma_G}}(\tilde{e})} Q$  is defined. As before, we denote by  $\tilde{\psi}_G: G \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G$  the canonical lift, into the fiber product group with respect to the surjection  $\pi_Q: G \rightarrow Q$ , of the crossed pair structure  $\tilde{\psi}: Q \rightarrow \text{Out}_{B^{\sigma_G}}(\tilde{e})$  on  $\tilde{e}$ . Define  $\chi_G: G \rightarrow \text{Out}((A, N, e, \vartheta), A)$  to be the combined homomorphism

$$\chi_G: G \xrightarrow{\tilde{\psi}_G} \text{Out}_{B^{\sigma_G}}(\tilde{e}) \xrightarrow{\gamma_b} \text{Out}((A, N, e, \vartheta), A). \quad (8.37)$$

Moreover, the composite homomorphism

$$\sigma_Q: Q \xrightarrow{\tilde{\psi}} \text{Out}_{B^{\sigma_G}}(\tilde{e}) \xrightarrow{\gamma_\#} \text{Out}^A(A, N, e, \vartheta)$$

yields a  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}^A(A, N, e, \vartheta)$  on the central  $S$ -algebra  $(A, N, e, \vartheta)$ . Denote by  $i: \Gamma \rightarrow U(A, N, e, \vartheta)$  the inclusion and by  $\tilde{\gamma}$  the combined homomorphism

$$\tilde{\gamma}: B^{\tilde{\psi}} \xrightarrow{\text{can}} \text{Aut}_{B^{\sigma_G}}(\tilde{e}) \xrightarrow{\gamma} \text{Aut}^A(A, N, e, \vartheta). \quad (8.38)$$

**Proposition 8.15.** *Write  $C = (A, N, e, \vartheta)$ . The homomorphisms  $\sigma_Q$ ,  $\kappa_G$ ,  $\chi_G$ ,  $\sigma_G$ ,  $i$ , and  $\tilde{\gamma}$  match in the following sense.*

(i) *The homomorphisms  $\sigma_Q$  and  $\chi_G$  yield a commutative diagram*

$$\begin{array}{ccccccc} e_{(T|S)}: 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\pi_Q} & Q \longrightarrow 1 \\ & & \downarrow \chi_N & & \downarrow \chi_G & & \downarrow \sigma_Q \\ e_{(A,C)}: 1 & \longrightarrow & N^{U(C)}(A)/U(A) & \longrightarrow & \text{Out}(C, A) & \longrightarrow & \text{Out}^A(C) \longrightarrow 1 \end{array} \quad (8.39)$$

*with exact rows.*

(ii) *The composite homomorphism*

$$G \xrightarrow{\chi_G} \text{Out}((A, N, e, \vartheta), A) \xrightarrow{\text{res}} \text{Out}(A) \quad (8.40)$$

*coincides with  $\sigma_G: G \rightarrow \text{Out}(A)$ .*

(iii) *The two homomorphisms  $i$  and  $\tilde{\gamma}$  yield a morphism of crossed 2-fold extensions*

$$\begin{array}{ccccccc} 1 & \longrightarrow & U(S) & \longrightarrow & \Gamma & \longrightarrow & B^{\tilde{\psi}} \longrightarrow Q \longrightarrow 1 \\ & & \parallel & & \downarrow i & & \downarrow \tilde{\gamma} \\ 1 & \longrightarrow & U(S) & \longrightarrow & U(A, N, e, \vartheta) & \longrightarrow & \text{Aut}^A(A, N, e, \vartheta) \longrightarrow \text{Out}^A(A, N, e, \vartheta) \longrightarrow 1 \end{array}$$

whence  $(i, \tilde{\gamma})$  induces a congruence  $(1, i, \cdot, 1): e_{\tilde{\psi}} \longrightarrow e_{((A, N, e, \vartheta), \sigma_Q)}$  of crossed 2-fold extensions.

(iv) The homomorphism  $\chi_N: N \rightarrow N^{U(C)}(A)/U(A)$  turns the embedding of  $A$  into  $C = (A, N, e, \vartheta)$  into a strong  $N$ -normal Deuring embedding with respect to  $\text{Id}: N \rightarrow \text{Aut}(T|S)$ .

*Proof.* (i) It is obvious that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi_Q} & Q \\ \tilde{\psi}_G \downarrow & & \tilde{\psi} \downarrow \\ \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G & \longrightarrow & \text{Out}_{B^{\sigma_G}}(\tilde{e}) \end{array} \quad (8.41)$$

is commutative. Combining this diagram with the commutative diagram (8.34), we obtain the right-hand square of (8.39). Since the lower row of that diagram is exact, the homomorphisms  $\chi_G$  and  $\sigma_Q$  induce the requisite homomorphism  $\chi_N: N \rightarrow N^{U(C)}/U(A)$ .

(ii) Consider the diagram

$$\begin{array}{ccccc} G & \xrightarrow{\tilde{\psi}_G} & \text{Out}_{B^{\sigma_G}}(\tilde{e}) \times_Q G & \xrightarrow{\gamma_b} & \text{Out}((A, N, e, \vartheta), A) \\ & \searrow \hat{\psi} & \downarrow \text{res}_{\#} & & \downarrow \text{res} \\ & & \text{Out}_{B^{\sigma_G}}(\hat{e}) & \xrightarrow{\gamma_{\#}} & \text{Out}(A) \end{array}$$

The right-hand square is commutative in an obvious manner. The left-hand triangle is commutative since, as noted earlier, the composite (8.30) coincides with  $\hat{\psi}$ . The upper row yields the homomorphism  $\chi_G: G \rightarrow \text{Out}((A, N, e, \vartheta), A)$ , by the very definition (8.37) of  $\chi_G$ .

As noted above, the composite (8.28), viz.  $G \xrightarrow{\hat{\psi}} \text{Out}_{B^{\sigma_G}}(\hat{e}) \xrightarrow{\hat{\gamma}_{\#}} \text{Out}(A)$ , yields the given  $G$ -normal structure  $\sigma_G: G \rightarrow \text{Out}(A)$  on  $A$ . Consequently (8.40) coincides with the structure map  $\sigma_G: G \rightarrow \text{Out}(A)$  as asserted.

(iii) This is obvious.

(iv) Consider the commutative diagram

$$\begin{array}{ccccccccc} e_{(T|S)}: 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \chi_N \downarrow & & \chi_G \downarrow & & \sigma_Q \downarrow & & \\ e_{(A,C)}: 1 & \longrightarrow & N^{U(C)}(A)/U(A) & \longrightarrow & \text{Out}(C, A) & \longrightarrow & \text{Out}^A(C) & \longrightarrow & 1 \\ & & \eta_b \downarrow & & \text{res} \downarrow & & \text{res} \downarrow & & \\ 1 & \longrightarrow & \text{Aut}(T|S) & \longrightarrow & \text{Aut}^S(T) & \xrightarrow{\text{res}} & \text{Aut}(S). & & \end{array}$$

By construction, the outer-most diagram coincides with the commutative diagram (6.1), and the left-most column is the composite (3.10), with  $N$  substituted for  $Q$  and  $\text{Aut}(T|S)$  for  $\text{Aut}(S)$ . Consequently the composite  $\eta_b \circ \chi_N: N \rightarrow \text{Aut}(T|S)$  is the identity. Since  $T|S$  is a Galois extension of commutative rings with Galois group  $N$ , by Proposition 4.3(ii), the algebra  $A$  coincides with the centralizer of  $T$  in  $C = (A, N, e, \vartheta)$  whence, by Proposition 3.11(iii), the homomorphism  $\eta_b$  is injective. Consequently  $\eta_b$  and  $\chi_N$  are isomorphisms, and  $\chi_N: N \rightarrow N^{U(C)}(A)/U(A)$  turns the embedding of  $A$  into  $C$  into a strong  $N$ -normal Deuring embedding with respect to  $\text{Id}: N \rightarrow \text{Aut}(T|S)$ .  $\square$

We can now complete the proof of Theorem 8.13: Since the structure homomorphism  $\kappa_G: G \rightarrow \text{Out}(A)$  is a  $G$ -normal structure relative to the action  $\kappa_G: G \rightarrow \text{Aut}^S(T)$  of  $G$  on  $T$ , by definition, the composite homomorphism  $G \xrightarrow{\sigma_G} \text{Out}(A) \xrightarrow{\text{res}} \text{Aut}^S(T)$  coincides with  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ ; since, by Proposition 8.15 (ii), (8.40) coincides with  $\sigma_G: G \rightarrow \text{Out}(A)$ , we conclude that the composite

$$G \xrightarrow{\chi_G} \text{Out}((A, N, e, \vartheta), A) \xrightarrow{\text{res}} \text{Aut}^S(T)$$

coincides with  $\kappa_G$ , cf. (8.11).

By Proposition 8.15(i), the diagram (8.39) is commutative, and by Proposition 8.15(iv), the homomorphism  $\chi_N: N \rightarrow N^{\text{U}(C)}(A)/\text{U}(A)$  turns the embedding of  $A$  into  $C$  into a strong  $N$ -normal Deuring embedding with respect to  $\text{Id}: N \rightarrow \text{Aut}(T|S)$ . Consequently, cf. (8.37), the homomorphism  $\chi_G: G \rightarrow \text{Out}((A, N, e, \vartheta), A)$  turns the embedding of  $A$  into  $(A, N, e, \vartheta)$  into a strong  $Q$ -normal Deuring embedding with respect to the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(A, N, e, \vartheta)$  on  $(A, N, e, \vartheta)$  and to the structure extension (8.1).

Proposition 8.15(ii) says that the  $G$ -normal structure  $G \rightarrow \text{Out}(A)$  on  $A$  associated to the strong  $Q$ -normal Deuring embedding, cf. (8.12), coincides with the given  $G$ -normal structure  $\sigma_G: G \rightarrow \text{Out}(A)$  on  $A$ .

Proposition 8.13 (i) and Proposition 8.15(iii) together entail that the  $Q$ -normal  $S$ -algebra  $((A, N, e, \vartheta), \sigma_Q)$  has Teichmüller class  $k$  as asserted since the crossed 2-fold extension  $e_{\tilde{\psi}}$  represents  $k$ .

The proof of Theorem 8.10 is now complete.  $\square$

*Proof of Complement 8.11.* This is an immediate consequence of Proposition 4.4(xi).  $\square$

Recall that  $B^{\tilde{\psi}}$  denotes the fiber product group  $B^{\tilde{\psi}} = \text{Aut}_{B^{\sigma_G}(\tilde{e})} \times_{\text{Out}_{B^{\sigma_G}(\tilde{e})} Q} Q$  with respect to the crossed pair structure  $\tilde{\psi}: Q \rightarrow \text{Out}_{B^{\sigma_G}(\tilde{e})}$  on  $\tilde{e}$ , and that, likewise,  $B^{A, \sigma_Q}$  denotes the fiber product group  $B^{A, \sigma_Q} = \text{Aut}^A(A, N, e, \vartheta) \times_{\text{Out}(A, N, e, \vartheta)} Q$  with respect to the  $Q$ -normal structure  $\sigma_Q: Q \rightarrow \text{Out}(A, N, e, \vartheta)$  on  $(A, N, e, \vartheta)$ .

**Complement 8.16.** *The canonical homomorphism  $B^{\tilde{\psi}} \rightarrow B^{A, \sigma_Q}$  induced by the action  $\tilde{\gamma}: B^{\tilde{\psi}} \rightarrow \text{Aut}^A(A, N, e, \vartheta)$  of  $B^{\tilde{\psi}}$  on the crossed product algebra  $(A, N, e, \vartheta)$ , cf. (8.38) above, and the surjection  $B^{\tilde{\psi}} \rightarrow Q$  is an isomorphism.*

*Proof.* The homomorphism  $B^{\tilde{\psi}} \rightarrow B^{A, \sigma_Q}$  makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{U}(S) & \longrightarrow & \Gamma & \longrightarrow & B^{\tilde{\psi}} & \longrightarrow & Q & \longrightarrow & 1 \\ & & \parallel & & \parallel & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \text{U}(S) & \longrightarrow & N^{\text{U}(A, N, e, \vartheta)}(A) & \longrightarrow & B^{A, \sigma_Q} & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

commutative whence the homomorphism  $B^{\tilde{\psi}} \rightarrow B^{A, \sigma_Q}$  is an isomorphism.  $\square$

## 9 Induced normal and equivariant structures

As before,  $S$  denotes a commutative ring and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of a group  $Q$  on  $S$ , and let  $R = S^Q$ . Recall that a *split algebra* over  $S$  is an algebra which is isomorphic



to an algebra of endomorphisms of some faithful  $S$ -module  $M$ . If  $S$  is a field, any split algebra admits an obvious  $Q$ -equivariant structure. However, if  $S$  is an arbitrary ring and  $M$  a faithful  $S$ -module, some more structure on  $M$  is necessary to guarantee the existence of a  $Q$ -equivariant structure on  $A = \text{End}_S(M)$  or at least of a  $Q$ -normal structure as we now explain.

### 9.1 Induced $Q$ -normal structures

Consider an  $S$ -module  $M$ . Suppose that  $M$  admits an  $S^t\Gamma$ -module structure for some group  $\Gamma$  which maps onto  $Q$  in such a way that  $\Gamma$  acts on the coefficients from  $S$  via the projection  $\pi: \Gamma \rightarrow Q$ , that is to say,  $\Gamma$  acts on  $M$  by semi-linear transformations in the sense that

$${}^x(sy) = (\pi({}^x s))({}^x y), \quad x \in \Gamma, s \in S, y \in M. \quad (9.1)$$

The group  $U(\text{End}_S(M))$  of invertible elements of  $\text{End}_S(M)$  coincides with the group  $\text{Aut}_S(M)$ . The action of  $\Gamma$  on  $M$  restricted to  $\ker(\pi)$  is an ordinary representation  $\alpha: \ker(\pi) \rightarrow \text{Aut}_S(M)$  on  $M$  by  $S$ -linear transformations. With the notation  $i: \ker(\pi) \rightarrow \Gamma$  for the inclusion, the action of  $\Gamma$  on  $M$  induces an action  $\beta: \Gamma \rightarrow \text{Aut}(\text{End}_S(M))$  of  $\Gamma$  on  $\text{End}_S(M)$  in such a way that

$$(\alpha, \beta): (\ker(\pi), \Gamma, i) \longrightarrow (U(\text{End}_S(M)), \text{Aut}(\text{End}_S(M)), \partial) \quad (9.2)$$

is a morphism of crossed modules; this morphism of crossed modules induces a  $Q$ -normal structure  $\sigma = \sigma_{(\alpha, \beta)}: Q \rightarrow \text{Out}(\text{End}_S(M))$  on  $\text{End}_S(M)$ , and we shall refer to such a structure as an *induced  $Q$ -normal structure* on  $\text{End}_S(M)$ . Accordingly we define an *induced  $Q$ -normal split algebra* to be a  $Q$ -normal algebra of the kind  $(\text{End}_S(M), \sigma)$  for some faithful  $S$ -module  $M$ , where  $\sigma$  is an induced  $Q$ -normal structure on  $\text{End}_S(M)$ .

Let  $M$  be an  $S$ -module. Define the subgroup  $\text{Aut}(M, Q)$  of  $\text{Aut}_R(M) \times Q$  by

$$\text{Aut}(M, Q) = \{(\alpha, x); \alpha(sy) = {}^x s \alpha(y), s \in S, y \in M\} \subseteq \text{Aut}_R(M) \times Q, \quad (9.3)$$

and let  $\pi^{\text{Aut}(M, Q)}: \text{Aut}(M, Q) \rightarrow Q$  denote the obvious homomorphism. The following is immediate, and we spell it out for later reference.

**Proposition 9.1.** (i) *The homomorphism  $\pi^{\text{Aut}(M, Q)}: \text{Aut}(M, Q) \rightarrow Q$  has  $\text{Aut}_S(M)$  as its kernel.*

(ii) *The homomorphism  $\pi^{\text{Aut}(M, Q)}$  is surjective if and only if  $M$  admits an  $S^t\Gamma$ -module structure for some group  $\Gamma$  which maps onto  $Q$  in such a way that  $\Gamma$  acts on the coefficients from  $S$  via the projection  $\pi: \Gamma \rightarrow Q$ , that is to say,  $\Gamma$  acts on  $M$  by semi-linear transformations in the sense that (9.1) holds. If this happens to be the case, the group  $\text{Aut}(M, Q)$  fits into a group extension*

$$1 \longrightarrow \text{Aut}_S(M) \longrightarrow \text{Aut}(M, Q) \xrightarrow{\pi^{\text{Aut}(M, Q)}} Q \longrightarrow 1. \quad (9.4)$$

(iii) *Suppose that the homomorphism  $\pi^{\text{Aut}(M, Q)}$  is surjective. Then the group extension (9.4) splits if and only if  $M$  admits an  $S^tQ$ -module structure, and the  $S^tQ$ -module structures on  $M$  are in bijective correspondence with the sections  $Q \rightarrow \text{Aut}(M, Q)$  for  $\pi^{\text{Aut}(M, Q)}$ .*

(iv) *In particular, a free  $S$ -module  $M$  admits an  $S^tQ$ -module structure and, if  $M$  has finite rank, the  $S^tQ$ -module structures on  $M$  are parametrized by  $S$ -bases.*

(v) *Suppose that  $M$  is a free  $S$ -module of finite rank. Then every  $Q$ -equivariant structure on  $\text{End}_S(M)$ , necessarily an induced one, arises from an  $S^tQ$ -module structure on  $M$ , and*

the  $Q$ -equivariant structures on the central  $S$ -algebra  $\text{End}_S(M)$  are parametrized by classes of  $S^t Q$ -module structures  $s: Q \rightarrow \text{Aut}(M, Q)$  (sections for  $\pi^{\text{Aut}(M, Q)}$ ) on  $M$ ; indeed, two sections  $s_1, s_2: Q \rightarrow \text{Aut}(M, Q)$  for  $\pi^{\text{Aut}(M, Q)}$  induce the same  $Q$ -equivariant structure on the  $S$ -algebra  $\text{End}_S(M)$  if and only if, relative to the obvious injection  $U(S) \rightarrow \text{Aut}_S(M)$ , the two sections  $s_1$  and  $s_2$  differ by a derivation  $Q \rightarrow U(S)$ . Furthermore,  $\text{End}_S(M)$  acquires a unique induced  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(\text{End}_S(M), Q)$ , necessarily an equivariant one.

*Proof.* We sketch an argument for statement (v). When  $M$  is free of finite rank, the homomorphism  $\beta: \text{Pic}(S) \rightarrow \text{Pic}(\text{End}_S(M))$  in the generalized Skolem-Noether theorem, i. e., in Proposition 3.1 above, is an isomorphism and hence every  $S$ -linear automorphism of  $\text{End}_S(M)$  is an inner automorphism. This observation implies the assertion. We leave the details to the reader.  $\square$

*Remark 9.2.* Proposition 9.1 (ii) shows that, in the definition of an induced  $Q$ -normal structure on a split algebra  $\text{End}_S(M)$  over some faithful  $S$ -module  $M$ , we may take the group  $\Gamma$  to be the group  $\text{Aut}(M, Q)$ .

**Proposition 9.3.** *Given an  $S$ -module  $M$ , the Teichmüller class  $[e_{(A, \sigma)}] \in H^3(Q, U(S))$  of an induced  $Q$ -normal structure  $\sigma: Q \rightarrow \text{Out}(A)$  on  $A = \text{End}_S(M)$  is zero.*

*Proof.* The  $Q$ -normal structure  $\sigma$  is induced by a semi-linear action of some group  $\Gamma$  on  $M$  that maps onto  $Q$  in such a way that  $\Gamma$  acts on the coefficients via this projection, combined with  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ . The corresponding morphism (9.2) of crossed modules induces a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \ker(\pi) & \longrightarrow & \Gamma & \xrightarrow{\pi} & Q \longrightarrow 1 \\ & & \alpha \downarrow & & \beta^\sigma \downarrow & & \parallel \\ & & U(A) & \xrightarrow{\partial^\sigma} & B^\sigma & \longrightarrow & Q \longrightarrow 1 \end{array}$$

with exact rows, cf. (3.8) for the notation  $\partial^\sigma$  etc. This implies that the Teichmüller class  $[e_{(A, \sigma)}] \in H^3(Q, U(S))$  is zero.  $\square$

## 9.2 Proof of the second assertion of Theorem 3.24

Suppose that the generalized  $Q$ -normal Azumaya  $S$ -algebra  $(A, \sigma_A)$  arises from an ordinary  $Q$ -normal algebra structure  $\sigma: Q \rightarrow \text{Out}(A, Q)$  on  $A$ . Then the construction of  $M_{\sigma_A}$  and  $C_{\sigma_A}$ , cf. Subsection 3.10 above, simplifies as follows: For each  $x \in Q$ , let  $\varepsilon_x: A \rightarrow A$  be an automorphism of grade  $x$  that lifts the automorphism  $\kappa_Q(x)$  of  $S$ , and take  $M_x$  to be the  $(A, A)$ -bimodule  $A_{\varepsilon(x)}$ , that is, the algebra  $A$  itself, viewed as an invertible  $(A, A)$ -bimodule of grade  $x$  via the structure map

$$A \otimes A_{\varepsilon(x)} \otimes A \longrightarrow A_{\varepsilon(x)}, \quad a_1 \cdot a \cdot a_2 = a_1 a(\varepsilon(x)(a_2)), \quad a, a_1, a_2 \in A.$$

Furthermore, given  $x, y \in Q$ , the isomorphism  $f_{x, y}: A_{\varepsilon(x)} \otimes_A A_{\varepsilon(y)} \rightarrow A_{\varepsilon(xy)}$  may be taken to be an inner automorphism of  $A$  in the sense that, for some invertible member  $u$  of  $A$ , given  $a_1 \otimes a_2 \in A_{\varepsilon(x)} \otimes_A A_{\varepsilon(y)}$ , the value  $f_{x, y}(a_1 \otimes a_2)$  equals  $ua_1 a_2 u^{-1}$ . Now  $M_{\sigma_A} = \bigoplus_{z \in Q} A_{\varepsilon(z)}$  is a free left  $A$ -module.

From the obvious isomorphisms

$$\text{End}_S(M_{\sigma_A}) = \prod_{u \in Q_A} \bigoplus_{v \in Q_A} \text{Hom}_S(M_u, M_v) \cong \prod_{u \in Q_A} \text{Hom}_S(M_u, M_{\sigma_A}) \quad (9.5)$$

we deduce that the canonical homomorphism  $A \otimes C \rightarrow \text{End}_S(M_{\sigma_A})$  is an isomorphism of  $S$ -algebras. (This is immediate when the group  $\kappa_Q(Q) \subseteq \text{Aut}(S)$  is finite.) Under this isomorphism, the tensor product  $Q$ -normal structure  $\sigma \otimes \sigma_C$  on  $A \otimes C$  corresponds to an induced  $Q$ -normal structure on  $\text{End}_S(M_{\sigma_A})$ . In view of Proposition 3.9, Proposition 9.3 entails that the sum  $[e_{(A, \sigma)}] + [e_{(C, \sigma_C)}] \in H^3(Q, U(S))$  is zero whence, in view of Proposition 3.7,  $[e_{(C^{\text{op}}, \sigma_{C^{\text{op}}})}] = [e_{(A, \sigma)}] \in H^3(Q, U(S))$  as asserted.

### 9.3 Induced $Q$ -equivariant structures

Consider an  $S$ -module  $M$ . Suppose that  $M$  admits an  $S^t\Gamma$ -module structure for some group  $\Gamma$  which maps onto  $Q$  in such a way that  $\Gamma$  acts on the coefficients from  $S$  via the projection  $\pi: \Gamma \rightarrow Q$ , consider the associated morphism (9.2) of crossed modules, and suppose that  $\alpha$  maps  $\ker(\pi)$  to  $U(S) \subseteq U(\text{End}_S(M))$ . Then the homomorphism  $\beta$  induces a  $Q$ -equivariant structure  $\tau = \tau_{(\alpha, \beta)}: Q \rightarrow \text{Aut}(\text{End}_S(M))$  on  $\text{End}_S(M)$ , and we shall refer to such a structure as an *induced  $Q$ -equivariant structure* on  $\text{End}_S(M)$ . Accordingly, we define an *induced  $Q$ -equivariant split algebra* to be a  $Q$ -equivariant algebra of the kind  $(\text{End}_S(M), \tau)$  for some faithful  $S$ -module  $M$ , where  $\tau$  is an induced  $Q$ -equivariant structure on  $\text{End}_S(M)$ .

Let  $M$  be an  $S$ -module let  $\tau: Q \rightarrow \text{Aut}(\text{End}_S(M))$  be a  $Q$ -equivariant structure on  $\text{End}_S(M)$ , let  $\text{Aut}(M, Q, \tau)$  denote the subgroup of  $\text{Aut}(M, Q)$  defined by

$$\text{Aut}(M, Q, \tau) = \{(\alpha, x); \alpha(ay) = (\tau(x)a)\alpha(y), a \in \text{End}_S(M), y \in M\} \subseteq \text{Aut}_R(M) \times Q,$$

and let  $\pi^{\text{Aut}(M, Q, \tau)}: \text{Aut}(M, Q, \tau) \rightarrow Q$  denote the canonical homomorphism. The following is immediate.

**Proposition 9.4.** (i) *The homomorphism  $\pi^{\text{Aut}(M, Q, \tau)}: \text{Aut}(M, Q, \tau) \rightarrow Q$  has  $U(S)$  as its kernel.*

(ii) *The homomorphism  $\pi^{\text{Aut}(M, Q, \tau)}$  is surjective if and only if  $M$  admits an  $S^t\Gamma$ -module structure for some group  $\Gamma$  which maps onto  $Q$  in such a way that  $\Gamma$  acts on  $M$  by semi-linear transformations via the projection  $\pi: \Gamma \rightarrow Q$  in the sense that (9.1) holds and such that  $\ker(\pi)$  maps to  $U(S)$ . If this happens to be the case, the group  $\text{Aut}(M, Q, \tau)$  fits into a group extension*

$$e^{M, \tau}: 1 \longrightarrow U(S) \longrightarrow \text{Aut}(M, Q, \tau) \xrightarrow{\pi^{\text{Aut}(M, Q, \tau)}} Q \longrightarrow 1. \quad (9.6)$$

**Remark 9.5.** Let  $M$  be a faithful  $S$ -module. Proposition 9.4 (ii) shows that, in the definition of an induced  $Q$ -equivariant structure on the split algebra  $\text{End}_S(M)$  over  $M$ , given a  $Q$ -equivariant structure  $\tau: Q \rightarrow \text{Aut}(\text{End}_S(M))$  on  $\text{End}_S(M)$ , we may take the group  $\Gamma$  to be the group  $\text{Aut}(M, Q, \tau)$ .

**Lemma 9.6.** *Given a faithful  $S$ -module  $M$ , let  $\tau: Q \rightarrow \text{Aut}(\text{End}_S(M))$  be a  $Q$ -equivariant structure on the central  $S$ -algebra  $\text{End}_S(M)$ , and suppose that  $\pi^{\text{Aut}(M, Q)}: \text{Aut}(M, Q) \rightarrow Q$  is surjective. Then  $\pi^{\text{Aut}(M, Q, \tau)}: \text{Aut}(M, Q, \tau) \rightarrow Q$  is surjective as well, whence  $\tau$  is then an induced  $Q$ -equivariant structure.*

*Proof.* By Proposition 9.1 (ii), the homomorphism  $\pi^{\text{Aut}(M, Q)}: \text{Aut}(M, Q) \rightarrow Q$  is surjective and induces, via the associated action  $\beta: \text{Aut}(M, Q) \rightarrow \text{Aut}(\text{End}_S(M))$  of  $\text{Aut}(M, Q)$  on  $\text{End}_S(M)$ , the  $Q$ -normal structure  $\sigma_\tau: Q \rightarrow \text{Out}(\text{End}_S(M))$  on  $\text{End}_S(M)$  associated to  $\tau$ .

The canonical homomorphism  $\text{Aut}(M, Q, \tau) \rightarrow Q$  is surjective. Indeed, let  $x \in \text{Aut}(M, Q)$ . Then the automorphisms  $\beta(x)$  and  $\tau(\pi^{\text{Aut}(M, Q)}(x))$  of  $\text{End}_S(M)$  have the same image in  $\text{Out}(\text{End}_S(M))$  whence there exists an  $S$ -linear automorphism  $\tilde{\alpha}_x$  of  $M$  so that, given  $a \in \text{End}_S(M)$ ,

$$\tilde{\alpha}_x(\beta(x)a)\tilde{\alpha}_x^{-1} = \tau(\pi(x))a.$$

Consequently, given  $q \in Q$  and a pre-image  $x \in \text{Aut}(M, Q)$  of  $q$ ,

$$\tilde{\alpha}_x(x(ay)) = (\tau(q)a)\tilde{\alpha}_x(x)y, \quad a \in \text{End}_S(M), y \in M;$$

thus the automorphism  $\alpha_x \in \text{Aut}_R(M)$  given by  $\alpha_x(y) = \tilde{\alpha}_x(x)y$  then yields a pre-image  $(\alpha_x, q) \in \text{Aut}(M, Q, \tau)$  of  $q \in Q$ . Hence the canonical homomorphism from  $\text{Aut}(M, Q, \tau)$  to  $Q$  is surjective. By construction, the induced action of  $\text{Aut}(M, Q, \tau)$  on  $\text{End}_S(M)$  induces the given  $Q$ -equivariant structure  $\tau$  on  $\text{End}_S(M)$ .  $\square$

**Corollary 9.7.** *Given a  $Q$ -equivariant structure  $\tau: Q \rightarrow \text{Aut}(\text{End}_S(M))$  on the central  $S$ -algebra  $\text{End}_S(M)$  over a faithful  $S$ -module  $M$ , suppose that the associated  $Q$ -normal structure  $\sigma_\tau: Q \rightarrow \text{Out}(\text{End}_S(M))$  on  $\text{End}_S(M)$  associated to  $\tau$  is induced as a  $Q$ -normal structure. Then the  $Q$ -equivariant structure  $\tau$  on  $\text{End}_S(M)$  is an induced  $Q$ -equivariant structure.*

## 9.4 Induced $Q$ -equivariant structures and crossed product algebras

Suppose that the group  $Q$  is a finite group. Let  $M$  be a faithful finitely generated projective  $S$ -module and  $\tau: Q \rightarrow \text{Aut}(\text{End}_S(M))$  an induced  $Q$ -equivariant structure on the split central  $S$ -algebra  $\text{End}_S(M)$ . With respect to an associated group extension  $e: U(S) \xrightarrow{i} \Gamma \xrightarrow{\pi^\Gamma} Q$  and morphism

$$(j, \beta): (U(S), \Gamma, i) \longrightarrow (\text{Aut}_S(M), \text{Aut}(\text{End}_S(M)), \partial)$$

of crossed modules inducing the  $Q$ -equivariant structure  $\tau$ , let  $M_e$  denote the  $S^t\Gamma$ -module that underlies the crossed product algebra  $(S, Q, e, \kappa_Q \circ \pi^\Gamma)$ , by construction, free as an  $S$ -module, let  $\tau_e: Q \rightarrow \text{Aut}(\text{End}_S(M_e))$  denote the associated induced  $Q$ -equivariant structure on  $\text{End}_S(M_e)$ , and consider the  $S$ -module  $\text{Hom}_S(M, M_e)$ , necessarily finitely generated projective and faithful. The association

$$\text{End}(M)^{\text{op}} \otimes \text{End}_S(M_e) \otimes \text{Hom}_S(M, M_e) \longrightarrow \text{Hom}_S(M, M_e) \quad (9.7)$$

which, for  $h \in \text{End}_S(M)$ ,  $f \in \text{End}_S(M_e)$ ,  $\varphi \in \text{Hom}_S(M, M_e)$ , is given by

$$h \otimes f \otimes \varphi \longmapsto f \circ \varphi \circ h,$$

identifies the central  $S$ -algebras  $\text{End}_S(M)^{\text{op}} \otimes \text{End}_S(M_e)$  and  $\text{End}_S(\text{Hom}_S(M, M_e))$ .

**Proposition 9.8.** *The diagonal action of  $\Gamma$  on  $\text{Hom}_S(M, M_e)$  given by the association*

$$(\alpha, \varphi) \longmapsto \alpha(\varphi) = \alpha \circ \varphi \circ \alpha^{-1}, \quad \alpha \in \Gamma, \quad \varphi \in \text{Hom}_S(M, M_e),$$

*is trivial on  $U(S) = \ker(\pi^\Gamma)$  and hence descends to an  $S^tQ$ -module structure*

$$Q \times \text{Hom}_S(M, M_e) \longrightarrow \text{Hom}_S(M, M_e) \quad (9.8)$$

on  $\text{Hom}_S(M, M_e)$ . Consequently, with the notation  $\tau_0$  for the  $Q$ -equivariant structure on  $(\text{End}_S(\text{Hom}_S(M, M_e)))$  induced by (9.8),

$$(\text{End}_S(M)^{\text{op}} \otimes \text{End}_S(M_e), \tau^{\text{op}} \otimes \tau_e) \cong (\text{End}_S(\text{Hom}_S(M, M_e)), \tau_0)$$

as  $Q$ -equivariant central  $S$ -algebras.

*Proof.* For both  $M$  and  $M_e$ , the restriction of the  $\Gamma$ -action to the kernel  $U(S)$  of  $\pi^\Gamma$  coincides with the  $U(S)$ -module structure coming from multiplication by members of the coefficient ring  $S$ . This implies the assertion.  $\square$

## 10 Crossed Brauer group, generalized crossed Brauer group, and Picard group

### 10.1 Crossed Brauer group

As before,  $S$  denotes a commutative ring and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of a group  $Q$  on  $S$ . Given a faithful finitely generated projective  $S$ -module, the central  $S$ -algebra  $\text{End}_S(M)$  is an Azumaya algebra. We will say that two  $Q$ -normal Azumaya  $S$ -algebras  $(A_1, \sigma_1)$  and  $(A_2, \sigma_2)$  are *normally Brauer equivalent* if there are faithful finitely generated projective  $S$ -modules  $M_1$  and  $M_2$  together with induced  $Q$ -normal structures

$$\rho_1: Q \rightarrow \text{Out}(B_1), \quad B_1 = \text{End}_S(M_1), \quad \rho_2: Q \rightarrow \text{Out}(B_2), \quad B_2 = \text{End}_S(M_2),$$

such that  $(A_1 \otimes B_1, \sigma_1 \otimes \rho_1)$  and  $(A_2 \otimes B_2, \sigma_2 \otimes \rho_2)$  are isomorphic  $Q$ -normal  $S$ -algebras. Since the tensor product of two induced  $Q$ -normal split algebras is again an induced  $Q$ -normal split algebra in an obvious manner, that relation, referred to henceforth as *normal Brauer equivalence*, is indeed an equivalence relation, similarly as in [AG60a, p. 381], and this equivalence relation is compatible with the operation of taking tensor products. Hence the equivalence classes constitute an abelian monoid; moreover, given a  $Q$ -normal Azumaya  $S$ -algebra  $(A, \sigma)$ , the map (1.1) being an isomorphism, the  $S$ -algebra  $(A \otimes A^{\text{op}}, \sigma \otimes \sigma^{\text{op}})$  is an induced  $Q$ -normal split algebra, the requisite semi-linear action on the  $S$ -module that underlies  $A$  being given by the action  $B^\sigma \rightarrow \text{Aut}(A)$  of the fiber product group  $B^\sigma = \text{Aut}(A) \times_Q \text{Out}(A)$  with respect to  $\sigma: Q \rightarrow \text{Out}(A)$  that yields the crossed 2-fold extension (3.8), and so, taking the class of  $(A^{\text{op}}, \sigma^{\text{op}})$  as the inverse of the class of  $(A, \sigma)$ , we get in fact an abelian group, the identity element being the equivalence class of induced  $Q$ -normal split algebras  $(\text{End}_S(M), \sigma)$  where  $M$  ranges over faithful finitely generated projective  $S$ -modules having the property that the obvious homomorphism  $\pi^{\text{Aut}(M, Q)}$  from  $\text{Aut}(M, Q)$  to  $Q$  is surjective, cf. Proposition 9.1 above. In particular,  $(S, \kappa_Q)$  represents the identity element. We refer to that group as the *crossed Brauer group* of  $S$  relative to  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ , denote this group by  $\text{XB}(S, Q)$ , and we refer to the construction just given as the *standard construction*. An equivalent construction of the crossed Brauer group (as the cokernel of a homomorphism of certain abelian monoids) is given in [FW71b, Theorem 4 p. 43], [FW00, Section 3, a few lines before Theorem 3.2] (written as  $QB(R, \Gamma)$ ), cf. (15.8) below.

**Theorem 10.1.** *The crossed Brauer group is a functor on the change of actions category *Change* introduced in Subsection 2.7 in such a way that the following hold.*

(i) The assignment to a  $Q$ -normal Azumaya  $S$ -algebra of its Teichmüller complex yields a natural homomorphism

$$t: \text{XB}(S, Q) \longrightarrow H^3(Q, U(S)) \quad (10.1)$$

of abelian groups.

(ii) The class  $[(A, \sigma)] \in \text{XB}(S, Q)$  of a  $Q$ -normal Azumaya  $S$ -algebra  $(A, \sigma)$  is zero if and only if, for some faithful finitely generated projective  $S$ -module  $M$ , the algebra  $A$  is isomorphic to  $\text{End}_S(M)$  in such a way that  $\sigma$  is an induced  $Q$ -normal structure.

*Proof.* Functoriality of the crossed Brauer group is a consequence of Proposition 3.10(i) together with the fact that induced structures on split algebras are preserved under change of rings. Statement (i) is a consequence of Proposition 9.3, combined with Propositions 3.7, 3.9 and 3.10(iii). We leave the details to the reader. As for Statement (ii), the “if” statement is manifest but the “only if” statement is not. We shall complete the proof thereof in the next subsection.  $\square$

*Remark 10.2.* A variant of the homomorphism (10.1), written there as  $\rho$ , is given in [FW00, Theorem 3.4 (i)] by the assignment to a  $Q$ -normal Azumaya  $S$ -algebra of an explicit 3-cocycle of  $Q$  with values in  $U(S)$ . The statement of Theorem 10.1 (ii) generalizes [AG60a, Proposition 5.3].

## 10.2 Crossed Brauer group and Picard group

Given a morphism  $(f, \varphi): (S, Q, \kappa) \rightarrow (T, G, \lambda)$  in the change of actions category *Change* introduced in Subsection 2.7, we denote by  $\text{XB}(T|S; G, Q)$  the kernel of the induced homomorphism from  $\text{XB}(S, Q)$  to  $\text{XB}(T, G)$ . Thus  $\text{XB}(S|S; Q, Q)$  is the trivial group whereas  $\text{XB}(S|S; \{e\}, Q)$  is the kernel of the forgetful homomorphism from  $\text{XB}(S, Q)$  to  $B(S)$ . The notation  $\text{XB}(\cdot; \cdot, \cdot)$  might look a bit heavy to the reader; we use it for the sake of consistency with a similar notation for the equivariant case to be introduced in Section 11 below.

We will now define a homomorphism from  $\text{XB}(S|S; \{e\}, Q)$  to  $H^1(Q, \text{Pic}(S))$ . To this end, let  $M$  be a faithful finitely generated projective  $S$ -module, and consider the split  $S$ -algebra  $\text{End}_S(M)$ . Given an algebra automorphism  $\alpha$  of  $\text{End}_S(M)$ , let  ${}^\alpha M$  denote the  $\text{End}_S(M)$ -module whose  $\text{End}_S(M)$ -module structure is given by the formula

$$a \cdot x = ({}^\alpha a)y, \quad a \in \text{End}_S(M), \quad y \in M;$$

in particular,  $S$  then acts on  ${}^\alpha M$  by

$$s \cdot y = ({}^\alpha|_S s)y, \quad s \in S, \quad y \in M,$$

and so the association  $a \mapsto {}^\alpha a$  yields an isomorphism  $\text{End}_S(M) \rightarrow \text{End}_S({}^\alpha M)$  of  $S$ -algebras. Consequently,  $J(\alpha) = \text{Hom}_{\text{End}_S(M)}({}^\alpha M, M)$  is a faithful finitely generated projective rank one  $S$ -module, cf., e. g., [RZ61, Lemma 9], and the evaluation map

$$\text{Hom}({}^\alpha M, M) \otimes_{\text{End}_S(M)} {}^\alpha M \longrightarrow M$$

is an isomorphism of  $S$ -modules [AG60b, Prop. A.6].

Now, let  $\sigma$  be a  $Q$ -normal structure on  $\text{End}_S(M)$ . Then  $(\text{End}_S(M), \sigma)$  represents a member of  $\text{XB}(S|S; \{e\}, Q)$ . Let  $w: Q \rightarrow \text{Aut}(\text{End}_S(M))$  be a morphism of the underlying sets which lifts  $\sigma$ .

**Lemma 10.3.** *Let  $w': Q \rightarrow \text{Out}(\text{End}_S(M))$  be another lifting of  $\sigma$ . Given  $q \in Q$ , there is an  $S$ -linear automorphism  $\alpha_q: M \rightarrow M$  so that*

$${}^{w(q)}a = \alpha_q({}^{w'(q)}a)\alpha_q^{-1}: M \longrightarrow M, \quad a \in \text{End}_S(M),$$

*whence the class  $[J(w(q))] \in \text{Pic}(S)$  depends only on  $M$  and  $\sigma$  and not on the choice of  $w$ . Hence the map*

$$d_{(M,\sigma)}: Q \longrightarrow \text{Pic}(S), \quad d_{(M,\sigma)}(q) = [J(w(q))] \in \text{Pic}(S), \quad q \in Q, \quad (10.2)$$

*is well-defined in the sense that it does not depend on the choice of  $w$ .*

*Proof.* This is an immediate consequence of the exactness of the sequence

$$\text{Aut}_S(M) \longrightarrow \text{Aut}(\text{End}_S(M)) \longrightarrow \text{Out}(\text{End}_S(M)) \longrightarrow 1.$$

□

The following is immediate.

**Proposition 10.4.** *Given  $q, r \in Q$ , the  $S$ -modules  $J(w(qr))$  and  $J(w(q)) \otimes {}^q J(w(r))$  are isomorphic in an obvious way. Consequently the map  $d_{(M,\sigma)}$  defined by (10.2) is a derivation on  $Q$  with values in  $\text{Pic}(S)$ .*

□

**Proposition 10.5.** *The class of  $d_{(M,\sigma)}$  in  $H^1(Q, \text{Pic}(S))$  depends only on  $(\text{End}_S(M), \sigma)$  and not on a particular choice of  $M$ .*

*Proof.* Let  $M'$  be another faithful finitely generated projective  $S$ -module so that  $\text{End}_S(M)$  and  $\text{End}_S(M')$  are isomorphic central  $S$ -algebras. Then  $J = \text{Hom}_{\text{End}_S(M)}(M, M')$  is a faithful finitely generated projective rank one  $S$ -module in such a way that  $M'$  and  $M \otimes J$  are isomorphic  $\text{End}_S(M)$ -modules under the obvious map, the  $\text{End}_S(M)$ -action on the latter being given by

$$a(y \otimes f) = ay \otimes f, \quad a \in \text{End}_S(M), \quad y \in M, \quad f \in J.$$

Given  $q \in Q$ , the module  $\text{Hom}_S({}^q J, S)$  represents  ${}^q [J]^{-1} \in \text{Pic}(S)$ , and so  $d_{(M,\sigma)}$  and  $d_{(M',\sigma)}$  differ by the inner derivation

$$Q \longrightarrow \text{Pic}(S), \quad q \longmapsto [J]({}^q [J]^{-1}), \quad q \in Q.$$

Hence the class of  $d_{(M,\sigma)}$  in  $H^1(Q, \text{Pic}(S))$  depends only on  $(\text{End}_S(M), \sigma)$  and not on the choice of  $M$ . □

**Proposition 10.6.** *Suppose that the  $Q$ -normal structure  $\sigma$  on  $\text{End}_S(M)$  is induced. Then  $d_{(M,\sigma)}$  is zero.*

*Proof.* Some semi-linear action of some group  $\Gamma$  on  $M$  maps onto  $Q$  via (say)  $\pi: \Gamma \rightarrow Q$  and yields an action  $\beta: \Gamma \rightarrow \text{Aut}(\text{End}_S(M))$  of  $\Gamma$  on  $\text{End}_S(M)$  which, in turn, induces  $\sigma$ . Let  $w' = Q \rightarrow \Gamma$  be a section for  $\pi$  of the underlying sets, and let  $w = \beta w'$ . Then each finitely generated projective rank one  $S$ -module  $J(w(q)) = \text{Hom}_{\text{End}_S(M)}({}^{w(q)}M, M)$  is isomorphic to  $S$ , and so  $d_{(M,\sigma)}$  is zero. □

**Proposition 10.7.** *Suppose that  $d_{(M,\sigma)}$  is an inner derivation, that is to say, there is a faithful finitely generated projective rank one  $S$ -module  $J$  so that, for each  $q \in Q$ , the  $S$ -modules  $J(w(q))$  and  ${}^qJ \otimes \text{Hom}_S(J, S)$  are isomorphic. Then  $(\text{End}_S(M), \sigma)$  is an induced  $Q$ -normal split algebra.*

*Proof.* For each  $q \in Q$ ,

$$\text{Hom}_{\text{End}_S(M)}({}^{w(q)}(M \otimes J), M \otimes J) \cong J(w(q)) \otimes \text{Hom}_S({}^qJ, S) \otimes J \cong S.$$

Hence, replacing  $M$  by  $M \otimes J$ , we may assume that  $\text{End}_S(M)$  has the property that each projective rank one  $S$ -module  $\text{Hom}_{\text{End}_S(M)}({}^{w(q)}M, M)$  is a free  $S$ -module  $Su_q$  on a single generator  $u_q$ , necessarily an isomorphism  $u_q: {}^{w(q)}M \rightarrow M$  of  $\text{End}_S(M)$ -modules. For each  $q \in Q$ , the isomorphism  $u_q$  then satisfies the identity

$$(u_q)^{-1}(ay) = {}^{w(q)}a(u_q)^{-1}(y), \quad a \in \text{End}_S(M), \quad y \in M.$$

Thus each  $q \in Q$  extends to a semi-linear transformation of  $M$ , and  $(\text{End}_S(M), \sigma)$  is therefore an induced  $Q$ -normal split algebra.  $\square$

Given a  $Q$ -normal Azumaya  $S$ -algebra  $(A, \sigma)$  that represents a member of  $\text{XB}(S|S; e, Q)$ , in view of normal Brauer equivalence, there are faithful finitely generated projective  $S$ -modules  $M_1$  and  $M_2$  together with an induced  $Q$ -normal structure  $\sigma_1: Q \rightarrow \text{Out}(\text{End}_S(M_1))$  on  $\text{End}_S(M_1)$  such that  $A \otimes \text{End}_S(M_1)$  and  $\text{End}_S(M_2)$  are isomorphic as central  $S$ -algebras and, under this isomorphism, the  $Q$ -normal structure  $\sigma \otimes \sigma_1$  on  $A \otimes \text{End}_S(M_1)$  corresponds to a  $Q$ -normal structure  $\sigma_2: Q \rightarrow \text{End}_S(M_2)$  on  $\text{End}_S(M_2)$ .

**Theorem 10.8.** *The assignment to the class  $[(A, \sigma)] \in \text{XB}(S|S; \{e\}, Q)$  of a  $Q$ -normal Azumaya  $S$ -algebra  $(A, \sigma)$  that represents a member of  $\text{XB}(S|S; e, Q)$  of  $[d_{M_2, \sigma_2}] \in H^1(Q, \text{Pic}(S))$  yields an injective homomorphism*

$$\iota: \text{XB}(S|S; \{e\}, Q) \longrightarrow H^1(Q, \text{Pic}(S)) \quad (10.3)$$

*which is, furthermore, natural on the change of actions category **Change** introduced in Subsection 2.7 above.*

*Proof.* Propositions 10.5 and 10.6 entail that (10.3) is well-defined, and Proposition 10.7 implies that the map (10.3) is injective. We leave the proofs that the map (10.3) is a natural homomorphism of abelian groups to the reader.  $\square$

*Remark 10.9.* The injectivity of (10.3) may be found in [FW00, Theorem 3.3].

*Proof of Theorem 10.1 (ii).* Suppose that  $[(A, \sigma)] = 0 \in \text{XB}(S, Q)$ . By [AG60a, Proposition 5.3], there is a faithful finitely generated projective  $S$ -module  $M$  such that  $A \cong \text{End}_S(M)$ , and  $[(\text{End}_S(M), \sigma)] = 0 \in \text{XB}(S|S; \{e\}, Q)$ . Then the derivation  $d_{(M,\sigma)}$  is an inner derivation whence, by Proposition 10.7,  $(\text{End}_S(M), \sigma)$  is an induced  $Q$ -normal split algebra.  $\square$



### 10.3 The generalized crossed Brauer group

Inspection shows that the assignment to a  $Q$ -normal Azumaya algebra  $(A, \sigma)$  of the associated generalized  $Q$ -normal Azumaya algebra  $(A, \Theta_\sigma)$  of  $\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$ , cf. (3.7), yields a homomorphism

$$\theta: \text{XB}(S, Q) \longrightarrow k\mathcal{R}ep(Q, \mathcal{B}_{S,Q}) \quad (10.4)$$

of abelian groups, cf. [FW00, Section 3, a few lines before Theorem 3.2; Theorem 3.3]. We will therefore refer to  $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$  as the *generalized crossed Brauer group* of  $S$  relative to  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ .

**Theorem 10.10.** (i) *With the notation can for the canonical injection, the diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{XB}(S|S; \{e\}, Q) & \longrightarrow & \text{XB}(S, Q) & \longrightarrow & \text{B}(S, Q) \\ & & \downarrow \iota & & \downarrow \theta & & \downarrow \text{can} \\ 0 & \longrightarrow & \text{H}^1(Q, \text{Pic}(S)) & \xrightarrow{j_{\mathcal{B}_{S,Q}}} & k\mathcal{R}ep(Q, \mathcal{B}_{S,Q}) & \xrightarrow{\mu_{\mathcal{B}_{S,Q}}} & \text{H}^0(Q, \text{B}(S)) \end{array} \quad (10.5)$$

*is a commutative diagram of abelian groups with exact rows whence, in particular, the homomorphism  $\theta$  is injective; here  $j_{\mathcal{B}_{S,Q}}$  is the homomorphism of abelian group (2.9) above for  $\mathcal{C}_Q = \mathcal{B}_{S,Q}$ , and  $\mu_{\mathcal{B}_{S,Q}}$  refers to the canonical homomorphism, cf., e. g., the sequence (2.14) above.*

(ii) *The composite of  $\theta$  with (3.23) coincides with (10.1).*

(iii) *If the group  $Q$  is finite, the homomorphism  $\theta$  is surjective and hence an isomorphism whence  $\iota$  is then an isomorphism as well.*

(iv) *If the group  $Q$  is finite, the homomorphism  $\text{can}: \text{B}(S, Q) \rightarrow \text{H}^0(Q, \text{B}(S))$  is the identity.*

In particular, when the group  $Q$  is a finite group, the exact sequence (2.14) is valid with  $\text{XB}(S, Q)$  substituted for  $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$ .

*Remark 10.11.* The injectivity of (10.4) may be found in [FW71b, Theorem 4 p. 43], [FW00, Theorem 3.3], but spelled out for monoids rather than groups. The commutative diagram (10.5) is essentially the diagram in [FW00, Theorem 3.3]. Statement (ii) above is equivalent to the statement of [FW00, Theorem 3.4 (iii)]. Statement (iii) above is equivalent to the statement of [FW00, Theorem 4.1 (ii)].

*Proof.* Commutativity of the diagram (10.5) is straightforward, and the injectivity of  $\iota$ , established in Theorem 10.8 above, entails that of  $\theta$ . Statement (ii) follows at once from Theorem 3.24. Statement (iii) is an immediate consequence of the fact that in case the group  $Q$  is finite, given the generalized  $Q$ -normal Azumaya  $S$ -algebra  $(A, \sigma)$ , the associated  $Q$ -normal algebra  $(C^{\text{op}}, \sigma_{C^{\text{op}}})$ , cf. Theorem 3.19, is a  $Q$ -normal Azumaya  $S$ -algebra that represents a member of  $\text{XB}(S, Q)$  which, under  $\theta$ , goes to the class of  $(A, \sigma)$  in  $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$ . Statement (iv) is the statement of Corollary 3.20.  $\square$

### 10.4 Behaviour under $Q$ -normal Galois extensions

Consider a  $Q$ -normal Galois extension  $T|S$  of commutative rings, with structure extension  $e_{(T|S)}: N \rightarrow G \xrightarrow{\pi_Q} Q$  and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ , cf. Section 6 above, and denote the injection of  $S$  into  $T$  by  $i: S \rightarrow T$ . Then the abelian group  $\text{XB}(T|S; G, Q)$  is defined relative to the associated morphism  $(i, \pi_Q): (S, Q, \kappa_Q) \rightarrow (T, G, \kappa_G)$  in the change of actions category *Change*, cf. (6.3) above.

**Theorem 10.12.** *The sequence*

$$\mathrm{XB}(T|S; G, Q) \xrightarrow{t} \mathrm{H}^3(Q, \mathrm{U}(S)) \xrightarrow{\inf} \mathrm{H}^3(G, \mathrm{U}(T))$$

*is exact and, furthermore, natural in the data. Moreover, each class in the image of  $t$  is also the Teichmüller class of some crossed pair algebra.*

*Proof.* The naturality of the constructions entails that  $\inf \circ t = 0$ . Moreover, by Theorem 7.5,  $\ker(\inf) \subset \mathrm{im}(t)$ , and each class in the image of  $t$  comes from some crossed pair algebra.  $\square$

Let  $\mathrm{Pic}(T|S)$  denote the kernel of the homomorphism  $\mathrm{Pic}(S) \rightarrow \mathrm{Pic}(T)$  induced by  $i: S \rightarrow T$ . Our next aim is to construct a homomorphism from  $\mathrm{H}^1(Q, \mathrm{Pic}(T|S))$  to  $\mathrm{XB}(T|S; G, Q)$ . To this end, view  $T$  as an  $S$ -module in the obvious way and let  $A = \mathrm{End}_S(T)$ . Now, given an automorphism  $\alpha$  of  $A$  so that  $\alpha|_S$  is the identity, as above we can turn  $T$  into a new  $A$ -module  ${}^\alpha T$  by means of  $\alpha$ , and  $J(\alpha) = \mathrm{Hom}_A({}^\alpha T, T)$  is a faithful finitely generated projective rank one  $S$ -module; since  $A \otimes T$  is a matrix algebra,  $J(\alpha)$  represents a member of  $\mathrm{Pic}(T|S)$ , and the association  $\alpha \mapsto [J(\alpha)]$  yields a homomorphism  $\mathrm{Aut}(A|S) \rightarrow \mathrm{Pic}(T|S)$  which we claim to be surjective. In order to justify this claim, we first observe that the obvious map  $j: T^t N \rightarrow A$ , as explained in Section 1, is an isomorphism, since  $T|S$  is a Galois extension of commutative rings with Galois group  $N$ . Now, given a derivation  $d: N \rightarrow \mathrm{U}(T)$ , define the automorphism  $\alpha_d$  of  $T^t N$  by

$$\alpha_d(tn) = d(n)tn, \quad t \in T, n \in N.$$

Then

$$\mathrm{Der}(N, \mathrm{U}(T)) \longrightarrow \mathrm{Aut}(T^t N|S), \quad d \longmapsto \alpha_d,$$

is a homomorphism, and  $[J(\alpha_d)] \in \mathrm{Pic}(T|S)$  is the image of  $[d] \in \mathrm{H}^1(N, \mathrm{U}(T))$  under the standard isomorphism  $\mathrm{H}^1(N, \mathrm{U}(T)) \rightarrow \mathrm{Pic}(T|S)$  (with  $N$  and  $T$  substituted for  $Q$  and  $S$ , respectively, this is, e.g., a consequence of the exactness of (12.1) below at the second term). Hence the homomorphism  $\mathrm{Aut}(A|S) \rightarrow \mathrm{Pic}(T|S)$  is surjective as asserted. Consequently the obvious homomorphism from  $\mathrm{Aut}(A|S)$  to  $\mathrm{Aut}(A, Q)$  fits into a commutative diagram

$$\begin{array}{ccccc} \mathrm{Aut}(A|S) & \longrightarrow & \mathrm{Pic}(T|S) & \xrightarrow{\cong} & \mathrm{Out}(A|S) \\ \downarrow & & \downarrow & & \\ \mathrm{Aut}(A, Q) & \longrightarrow & \mathrm{Out}(A, Q) & & \end{array}$$

where the horizontal maps are surjective. Since the  $G$ -action on  $T$  and that on  $N$  induce a canonical section  $\sigma_0: Q \rightarrow \mathrm{Out}(A, Q)$ , the canonical homomorphism  $\mathrm{Out}(A, Q) \rightarrow Q$  is surjective as well. Consequently the sequence

$$0 \longrightarrow \mathrm{Pic}(T|S) \longrightarrow \mathrm{Out}(A, Q) \longrightarrow Q \longrightarrow 1$$

is exact. Now, given a derivation  $d: Q \rightarrow \mathrm{Pic}(T|S)$ , define the homomorphism

$$\sigma_d: Q \longrightarrow \mathrm{Out}(A, Q)$$

by  $\sigma(q) = d(q)\sigma_0(q)$ , as  $q$  ranges over  $Q$ . Then  $(A, \sigma_d)$  is a  $Q$ -normal Azumaya  $S$ -algebra.

We mention without proof the following.

**Theorem 10.13.** *The association  $d \mapsto (\text{End}_S(T), \sigma_d)$ , as  $d$  ranges over derivations from  $Q$  to  $\text{Pic}(T|S)$ , yields a natural isomorphism*

$$H^1(Q, \text{Pic}(T|S)) \longrightarrow \text{XB}(S|S; \{e\}, Q) \cap \text{XB}(T|S; G, Q) \quad (10.6)$$

*of abelian groups in such a way that the resulting sequence*

$$0 \longrightarrow H^1(Q, \text{Pic}(T|S)) \longrightarrow \text{XB}(T|S; G, Q) \longrightarrow H^0(Q, \text{B}(T|S)) \quad (10.7)$$

*is exact.*

## 11 The equivariant Brauer group

As before,  $S$  denotes a commutative ring and  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  an action of a group  $Q$  on  $S$ .

### 11.1 The construction

Given a split algebra  $\text{End}_S(M)$  for some faithful  $S$ -module  $M$ , we will refer to an induced  $Q$ -equivariant structure  $\phi: Q \rightarrow \text{Aut}(\text{End}_S(M))$  on  $\text{End}_S(M)$  that arises from an  $S^tQ$ -module structure on  $M$  as a *trivially induced  $Q$ -equivariant structure*, and we then refer to  $(\text{End}_S(M), \phi)$  as a *trivially induced  $Q$ -equivariant split algebra*. We will say that two  $Q$ -equivariant Azumaya  $S$ -algebras  $(A_1, \tau_1)$  and  $(A_2, \tau_2)$  are *equivariantly Brauer equivalent* if there are  $S^tQ$ -modules  $M_1$  and  $M_2$  whose underlying  $S$ -modules are faithful finitely generated projective such that, relative to the associated  $Q$ -equivariant structures

$$\phi_1: Q \rightarrow \text{Aut}(B_1), \quad B_1 = \text{End}_S(M_1), \quad \phi_2: Q \rightarrow \text{Aut}(B_2), \quad B_2 = \text{End}_S(M_2),$$

the algebras  $(A_1 \otimes B_1, \tau_1 \otimes \phi_1)$  and  $(A_2 \otimes B_2, \tau_2 \otimes \phi_2)$  are isomorphic  $Q$ -equivariant  $S$ -algebras. Since the tensor product of two trivially induced  $Q$ -equivariant split algebras is again a trivially induced  $Q$ -equivariant split algebra in an obvious manner, that relation is an equivalence relation, referred to henceforth as *equivariant Brauer equivalence*, cf. Section 10 above or [AG60a, p. 381]; under the operation of tensor product and under the assignment to the class of an equivariant algebra  $(A, \tau)$  of the class of  $(A^{\text{op}}, \tau^{\text{op}})$ , the equivalence classes constitute an abelian group, the identity element being the equivalence class of trivially induced  $Q$ -equivariant split algebras  $(\text{End}_S(M), \tau)$  where  $M$  ranges over  $S^tQ$ -modules whose underlying  $S$ -module is faithful and finitely generated projective. This group is the *equivariant Brauer group* of  $S$  with respect to  $\kappa_Q: Q \rightarrow \text{Aut}(S)$ , introduced in [FW71a] and [FW71b, p. 40]. We will denote this group by  $\text{EB}(S, Q)$ , and we refer to the construction just given as the *standard construction*.

### 11.2 Some properties of the equivariant Brauer group

Let  $R = S^Q$ . It is manifest that extension of scalars yields an obvious homomorphism  $\text{B}(R) \rightarrow \text{EB}(S, Q)$ . If  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ , by Galois descent (1.2) (ii), that homomorphism is actually an isomorphism.

In the general case of an arbitrary action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  of  $Q$  on  $S$ , the following holds.

**Theorem 11.1.** (i) *The equivariant Brauer group is a functor on the change of actions category  $\mathcal{Change}$  introduced in Subsection 2.7.*

(ii) *The assignment to a  $Q$ -equivariant Azumaya  $S$ -algebra of its canonically associated  $Q$ -normal  $S$ -algebra yields a natural homomorphism*

$$\text{res}: \text{EB}(S, Q) \longrightarrow \text{XB}(S, Q).$$

(iii) *The composite*

$$\text{EB}(S, Q) \xrightarrow{\text{res}} \text{XB}(S, Q) \xrightarrow{t} \text{H}^3(Q, \text{U}(S)) \quad (11.1)$$

*is zero. If furthermore, the group  $Q$  is finite, the sequence (11.1) is exact and, furthermore, necessarily natural in the data.*

*Proof.* This follows from the observation that induced equivariant structures on split algebras are preserved under change of rings, together with Proposition 3.10(ii) and Theorem 5.1; we only note that in case that  $Q$  is a finite group the matrix algebra  $\text{M}_{|Q|}(A)$  over an Azumaya  $S$ -algebra is again an Azumaya  $S$ -algebra.  $\square$

Given a morphism  $(f, \varphi): (S, Q, \kappa) \rightarrow (T, G, \lambda)$  in the change of actions category  $\mathcal{Change}$  introduced Subsection 2.7, we denote by  $\text{EB}(T|S; G, Q)$  the kernel of the combined map

$$\text{EB}(S, Q) \longrightarrow \text{XB}(S, Q) \longrightarrow \text{XB}(T, G);$$

this kernel  $\text{EB}(T|S; G, Q)$  is the subgroup of  $\text{EB}(S, Q)$  that consists of classes of  $Q$ -equivariant  $S$ -algebras  $(A, \tau)$  so that  $(A \otimes T, \tau_{(f, \varphi)})$  is an induced  $G$ -normal split algebra and hence, in view of Proposition 9.7, an induced  $G$ -equivariant split algebra; see Proposition 3.10 (ii) for the notation  $\tau_{(f, \varphi)}$ . Thus, in particular,  $\text{EB}(S|S; Q, Q)$  is the kernel of the canonical homomorphism from  $\text{EB}(S, Q)$  to  $\text{XB}(S, Q)$  whereas  $\text{EB}(S|S; \{e\}, Q)$  is the kernel of the forgetful homomorphism from  $\text{EB}(S, Q)$  to  $\text{B}(S)$ . It is obvious that the restriction homomorphism  $\text{res}$  from  $\text{EB}(S, Q)$  to  $\text{XB}(S, Q)$  induces a homomorphism  $\text{res}: \text{EB}(T|S; G, Q) \longrightarrow \text{XB}(T|S; G, Q)$ .

Consider a  $Q$ -normal Galois extension  $T|S$  of commutative rings, with structure extension  $e_{(T|S)}: N \rightarrowtail G \xrightarrow{\pi_Q} Q$  and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ , cf. Section 6 above, and denote the injection of  $S$  into  $T$  by  $i: S \rightarrow T$ . Then the abelian groups  $\text{EB}(T|S; G, Q)$  and  $\text{XB}(T|S; G, Q)$  are defined relative to the morphism  $(i, \pi_Q): (S, Q, \kappa_Q) \longrightarrow (T, G, \kappa_G)$  in the change of actions category  $\mathcal{Change}$  associated to the data, cf. (6.3) above.

**Theorem 11.2.** *Suppose that  $Q$  is a finite group. Then the sequence*

$$\text{EB}(T|S; G, Q) \xrightarrow{\text{res}} \text{XB}(T|S; G, Q) \xrightarrow{t} \text{H}^3(Q, \text{U}(S)) \xrightarrow{\text{inf}} \text{H}^3(G, \text{U}(T)) \quad (11.2)$$

*is exact and natural.*

*Proof.* The statement of the theorem is a consequence of Theorems 5.1, 8.9, 10.1 (ii), 10.12, and 11.1. For if  $(A, \sigma)$  represents a member of  $\text{XB}(T|S; G, Q)$  with zero Teichmüller class, by Theorem 5.1, we may assume  $(A, \sigma)$  to be equivariant, i.e.,  $\sigma = \sigma_\tau$  for some equivariant structure  $\tau$ . Now the  $G$ -normal algebra  $(A \otimes T, \sigma_{(i, \pi_G)})$  represents zero in  $\text{XB}(T, G)$  and hence is an induced  $G$ -normal split algebra, by Theorem 10.1 (ii). By Proposition 9.7,  $(A \otimes T, \tau_{(i, \pi_G)})$  is an induced  $G$ -equivariant split algebra.  $\square$

### 11.3 Group extensions and equivariant Azumaya algebras

We continue to exploit the  $Q$ -normal Galois extension spelled out in the previous subsection.

Let  $R = S^Q$ , let  $e_G: U(T) \xrightarrow{i_{e_G}} \Gamma_G \xrightarrow{\pi_{e_G}} G$  be a group extension, and denote the restriction to  $N$  of the group extension  $e_G$  by  $e_N: U(T) \xrightarrow{i_{e_N}} \Gamma_N \xrightarrow{\pi_{e_N}} N$ . Then the crossed product  $S$ -algebra  $A = (T, N, e_N, \pi_{e_N})$  and the crossed product  $R$ -algebra  $(T, G, e_G, \kappa_G \circ \pi_{e_G})$  are defined, the former being an Azumaya  $S$ -algebra, since  $T|S$  is a Galois extension of commutative rings with Galois group  $N$  (cf. Proposition 4.4(xi)), and  $A$  is a subalgebra of  $(T, G, e_G, \kappa_G \circ \pi_{e_G})$ . Consider the resulting group extension  $e_Q: \Gamma_N \xrightarrow{j_{e_Q}} \Gamma_G \xrightarrow{\pi_{e_Q}} Q$ , of the kind (4.1) above. Conjugation in  $\Gamma_G$  induces an action  $\vartheta_{e_G}: \Gamma_G \rightarrow \text{Aut}(A)$  of  $\Gamma_G$  on  $A$  such that, with the notation  $i^{\Gamma_N}: \Gamma_N \rightarrow U(A)$  for the obvious injection, the pair  $(i^{\Gamma_N}, \vartheta_{e_G})$  is a morphism  $(\Gamma_N, \Gamma_G, j_{e_Q}) \rightarrow (U(A), \text{Aut}(A), \partial)$  of crossed modules of the kind (4.2), and this morphism, in turn, induces a  $Q$ -normal structure  $\sigma_{\vartheta_{e_G}}: Q \rightarrow \text{Out}(A)$  on  $A$ ; thus the crossed product  $R$ -algebra of  $(T, G, e_G, \kappa_G \circ \pi_{e_G})$  can now be written as the crossed product  $R$ -algebra  $(A, Q, e_Q, \vartheta_{e_G})$  relative to the group extension  $e_Q$  and the morphism  $(i^{\Gamma_N}, \vartheta_{e_G})$  of crossed modules, cf. Section 4 above. In particular, the left  $A$ -module  $M_{e_Q}$  that underlies the algebra  $(T, G, e_G, \kappa_G \circ \pi_{e_G}) \cong (A, Q, e_Q, \vartheta_{e_G})$  is free with basis in one-one correspondence with the elements of  $Q$ , and the  $Q$ -equivariant structure (4.5), viz.  $\tau_{e_Q}: Q \rightarrow \text{Aut}({}_A \text{End}(M_{e_Q}))$ , is defined. When the group  $Q$  is finite, the algebra  ${}_A \text{End}(M_{e_Q})$  is an Azumaya  $S$ -algebra.

**Proposition 11.3.** *Suppose that the group  $Q$  is finite. Then the assignment to a group extension  $e_G$  of  $G$  by  $U(T)$  of the  $Q$ -equivariant algebra  $({}_A \text{End}(M_{e_Q})^{\text{op}}, \tau_{e_Q}^{\text{op}})$  yields a homomorphism*

$$\text{cpr}: H^2(G, U(T)) \longrightarrow \text{EB}(T|S; G, Q) \quad (11.3)$$

*of abelian groups that is natural on the change of actions category **Change**. In the special case where  $T = S$  and  $N$  is the trivial group, the homomorphism (11.3) takes the form*

$$\text{cpr}: H^2(Q, U(S)) \longrightarrow \text{EB}(S|S; Q, Q). \quad \square \quad (11.4)$$

## 12 The seven and eight term exact sequences

Suppose that the group  $Q$  is finite. With a slight abuse of notation, we will denote by  $\text{cpr}: H^2(Q, U(S)) \rightarrow \text{EB}(S, Q)$  the composite of (11.4) with the injection of  $\text{EB}(S|S; Q, Q)$  into  $\text{EB}(S, Q)$  as well.

**Theorem 12.1.** *The group  $Q$  being finite, the extension*

$$\dots \longrightarrow (\text{Pic}(S))^Q \xrightarrow{\omega_{\text{Pic}, S, Q}} H^2(Q, U(S)) \xrightarrow{\text{cpr}} \text{EB}(S, Q) \xrightarrow{\text{res}} \text{XB}(S, Q) \xrightarrow{t} H^3(Q, U(S)) \quad (12.1)$$

*of the exact sequence (2.15) is defined and yields a seven term exact sequence that is natural in terms of the data. If, furthermore,  $S|R$  is a Galois extension of commutative rings over  $R = S^Q$  with group  $Q$ , then, with  $\text{Pic}(S|R)$ ,  $\text{Pic}(R)$  and  $B(R)$  substituted for, respectively  $H^1(Q, U(S))$ ,  $\text{EPic}(S, Q)$  and  $\text{EB}(S, Q)$ , the homomorphisms  $\text{cpr}$  and  $\text{res}$  being modified accordingly, the sequence is exact as well.*

**Remark 12.2.** The lower long sequence in [FW00, Theorem 4.2] yields a long exact sequence of the kind (12.1).

*Proof.* The *exactness at*  $\text{XB}(S, Q)$  follows from Theorem 5.1 or Theorem 11.1.

*Exactness at*  $\text{H}^2(Q, \text{U}(S))$ : Let  $J$  be a finitely generated projective rank one  $S$ -module representing a class in  $(\text{Pic}(S))^Q$ , consider the group  $\text{Aut}(J, Q) \cong \text{Aut}_{\mathcal{P}\text{ic}_{S,Q}}(J)$ , i. e., the group (9.3) with  $J$  substituted for  $M$ , and let

$$e_J: 0 \longrightarrow \text{U}(S) \longrightarrow \text{Aut}(J, Q) \xrightarrow{\pi_J} Q \longrightarrow 1 \quad (12.2)$$

be the associated group extension (2.6) with  $\mathcal{P}\text{ic}_{S,Q}$  substituted for  $\mathcal{C}_Q$  so that  $\omega_{\mathcal{P}\text{ic}_{S,Q}}[J] = [e_J] \in \text{H}^2(Q, \text{U}(S))$ . Consider the crossed product  $R$ -algebra  $(S, Q, e_J, \kappa_Q \circ \pi_J)$ , let  $M_{e_J}$  denote the free  $S$ -module that underlies  $(S, Q, e_J, \kappa_Q \circ \pi_J)$ , and recall that the corresponding association (4.4), now of the kind  $\text{Aut}(J, Q) \times M_{e_J} \rightarrow M_{e_J}$ , induces, on the Azumaya  $S$ -algebra  $\text{End}_S(M_{e_J})$ , via the association

$$(\alpha, f) \longmapsto \alpha(f) = \alpha \circ f \circ \alpha^{-1}, \quad \alpha \in \text{Aut}(J, Q), \quad f \in \text{End}_S(M_{e_J}),$$

an induced  $Q$ -equivariant structure  $\tau_{e_J}: Q \rightarrow \text{Aut}(\text{End}_S(M_{e_J}))$  where we do not distinguish in notation between  $\alpha \in \text{Aut}(J, Q)$  and the associated semi-linear automorphism  $\alpha: M_{e_J} \rightarrow M_{e_J}$ . By construction, then, the value  $\text{cpr}([e_J]) \in \text{EB}(S|S; Q, Q) \subseteq \text{EB}(S, Q)$  is represented by the  $Q$ -equivariant Azumaya algebra  $(\text{End}(M_{e_J})^{\text{op}}, \tau_{e_J}^{\text{op}})$ .

Since  $J$  is a faithful finitely generated projective rank one  $S$ -module, the operation of composition  $(f, \varphi) \longmapsto f_{\#}(\varphi) = f \circ \varphi$ , as  $f$  ranges over  $\text{End}(M_{e_J})$  and  $\varphi$  over  $\text{Hom}_S(J, M_{e_J})$ , induces an isomorphism

$$\text{End}_S(M_{e_J}) \longrightarrow \text{End}_S(\text{Hom}_S(J, M_{e_J})), \quad f \longmapsto f_{\#}, \quad (12.3)$$

of  $S$ -algebras. Since the restriction of the  $\text{Aut}(J, Q)$ -action on  $M_{e_J}$  to the kernel  $\text{U}(S)$  of  $\pi_J$  coincides with the  $\text{U}(S)$ -module structure coming from multiplication by members of the coefficient ring  $S$ , the diagonal action of  $\text{Aut}(J, Q)$  on  $\text{Hom}_S(J, M_{e_J})$  given by the assignment to  $(\alpha, \varphi)$  of  $\alpha_J(\varphi) = \alpha \circ \varphi \circ \alpha^{-1}$ , where  $\alpha \in \text{Aut}(J, Q)$  and  $\varphi \in \text{Hom}_S(J, M_{e_J})$ , descends to an action

$$Q \times \text{Hom}_S(J, M_{e_J}) \longrightarrow \text{Hom}_S(J, M_{e_J}) \quad (12.4)$$

of  $Q$  on  $\text{Hom}_S(J, M_{e_J})$  that turns  $\text{Hom}_S(J, M_{e_J})$  into an  $S^t Q$ -module. The  $S^t Q$ -module structure (12.4) on  $\text{Hom}_S(J, M_{e_J})$ , in turn, induces a trivially induced  $Q$ -equivariant structure on  $\text{End}_S(\text{Hom}_S(J, M_{e_J}))$  via the association

$$(\alpha, h) \longmapsto \alpha_J(h) = \alpha_J \circ h \circ \alpha_J^{-1}, \quad \alpha \in \text{Aut}(J, Q), \quad h \in \text{End}_S(\text{Hom}_S(J, M_{e_J})).$$

By construction, given  $\varphi \in \text{Hom}_S(J, M_{e_J})$ ,  $f \in \text{End}_S(M_{e_J})$ , and  $\alpha \in \text{Aut}(J, Q)$ ,

$$\begin{aligned} (\alpha_J(f_{\#}))(\varphi) &= (\alpha_J \circ f_{\#} \circ \alpha_J^{-1})(\varphi) = \alpha_J(f_{\#}(\alpha_J^{-1}(\varphi))) = \alpha_J(f_{\#}(\alpha^{-1} \circ \varphi \circ \alpha)) \\ &= \alpha_J(f \circ \alpha^{-1} \circ \varphi \circ \alpha) = \alpha \circ (f \circ \alpha^{-1} \circ \varphi \circ \alpha) \alpha^{-1} = \alpha \circ f \circ \alpha^{-1} \circ \varphi \end{aligned}$$

whence the induced  $Q$ -equivariant structure  $\tau_{e_J}: Q \rightarrow \text{Aut}(\text{End}_S(M_{e_J}))$  gets identified, under the isomorphism (12.3), with the trivially induced  $Q$ -equivariant structure induced by the  $S^t Q$ -module structure (12.4) on  $\text{Hom}_S(J, M_{e_J})$ . Consequently  $(\text{End}_S(M_{e_J}), \tau_{e_J})$  represents zero in  $\text{EB}(S, Q)$ .

Conversely, let  $e: \text{U}(S) \hookrightarrow \Gamma \xrightarrow{\pi} Q$  be a group extension, consider the associated crossed product algebra  $(S, Q, e, \kappa_Q \circ \pi)$ , let  $M_e$  denote the free  $S$ -module that underlies the algebra  $(S, Q, e, \kappa_Q \circ \pi)$ , let  $\tau_e: Q \rightarrow \text{Aut}(\text{End}_S(M_e))$  denote the corresponding induced  $Q$ -equivariant structure, and suppose that  $(\text{End}_S(M_e), \tau_e)$  represents zero in  $\text{EB}(S, Q)$ . In view

of equivariant Brauer equivalence, there are faithful finitely generated projective  $S$ -modules  $M_1$  and  $M_2$  which admit, furthermore,  $S^t Q$ -module structures so that, with the notation  $\tau_1: Q \rightarrow \text{Aut}(\text{End}_S(M_1))$  and  $\tau_2: Q \rightarrow \text{Aut}(\text{End}_S(M_2))$  for the associated trivially induced  $Q$ -equivariant structures,  $(\text{End}_S(M_e), \tau_e) \otimes (\text{End}_S(M_1), \tau_1)$  and  $(\text{End}_S(M_2), \tau_2)$  are isomorphic as  $Q$ -equivariant  $S$ -algebras. Let  $J = \text{Hom}_{\text{End}_S(M_e \otimes M_1)}(M_e \otimes M_1, M_2)$ ; this is a faithful finitely generated projective rank one  $S$ -module. Moreover, the association

$$\Gamma \times J \longrightarrow J, (x, f) \mapsto \pi(x)fx^{-1}: M_e \otimes M_1 \longrightarrow M_2, x \in \Gamma, f \in J,$$

where we do not distinguish in notation between  $x \in \Gamma$  and the induced automorphisms of  $M_e \otimes M_1$  nor between  $\pi(x) \in Q$  and the induced automorphism of  $M_2$ , yields a semi-linear action of  $\Gamma$  on  $J$ ; hence the group extension  $e_J$ , cf. (12.2), is defined relative to  $J$ , and the  $\Gamma$ -action on  $J$ , in turn, induces a homomorphism  $\Gamma \rightarrow \text{Aut}(J, Q)$  and hence yields a congruence  $(1, \cdot, 1): e \rightarrow e_J$  of group extensions; in particular,  $[J] \in (\text{Pic}(S))^Q$ . The congruence  $(1, \cdot, 1): e \rightarrow e_J$  of group extensions entails that  $\omega_{\text{Pic}_S, Q}[J] = [e] \in H^2(Q, U(S))$ .

*Exactness at  $\text{EB}(S, Q)$ :* Since for a group extension  $e$  of  $U(S)$  by  $Q$  the  $Q$ -equivariant structure  $\tau_e$  on the algebra  $\text{End}_S(M_e)$  of  $S$ -linear endomorphisms of the free  $S$ -module  $M_e$  that underlies the corresponding crossed product algebra  $(S, Q, e, \kappa_Q \circ \pi)$  is an induced  $Q$ -equivariant structure, it is obvious that the composite  $\text{res} \circ \text{cpr}$  is zero.

Now we will show that  $\ker(\text{res}) \subset \text{im}(\text{cpr})$ . Thus, let  $(A, \tau)$  be a  $Q$ -equivariant Azumaya  $S$ -algebra, and suppose that the class of its associated  $Q$ -normal algebra  $(A, \sigma_\tau)$  goes to zero in  $\text{XB}(S, Q)$ . In view of normal Brauer equivalence, there are induced  $Q$ -normal split algebras  $(\text{End}_S(M_1), \sigma_1)$  and  $(\text{End}_S(M_2), \sigma_2)$  such that  $(A, \sigma_\tau) \otimes (\text{End}_S(M_1), \sigma_1)$  and  $(\text{End}_S(M_2), \sigma_2)$  are isomorphic as  $Q$ -normal  $S$ -algebras. Replacing  $M_1$  and  $M_2$  by  $S^{[Q]} \otimes M_1$  and  $S^{[Q]} \otimes M_2$ , respectively, and adjusting the notation accordingly, since  $(\text{End}_S(M_1), \sigma_1)$  has zero Teichmüller class, by Theorem 5.1, the  $Q$ -normal structure  $\sigma_1$  on  $\text{End}_S(M_1)$  lifts to a  $Q$ -equivariant structure  $\tau_1: Q \rightarrow \text{End}_S(M_1)$  on  $\text{End}_S(M_1)$  such that  $\sigma_1 = \sigma_{\tau_1}$ . Thus  $(A \otimes \text{End}_S(M_1), \sigma_{\tau \otimes \tau_1})$  and  $(\text{End}_S(M_2), \sigma_2)$  are isomorphic as  $Q$ -normal  $S$ -algebras. Since  $\sigma_{\tau \otimes \tau_1}$  is equivariant, so is  $\sigma_2$ ; more precisely, the  $Q$ -equivariant structure  $\tau \otimes \tau_1$  induces, on  $\text{End}_S(M_2)$ , via the isomorphism  $A \otimes \text{End}_S(M_1) \cong \text{End}_S(M_2)$ , a  $Q$ -equivariant structure  $\tau_2: Q \rightarrow \text{End}_S(M_2)$  such that  $(A \otimes \text{End}_S(M_1), \tau \otimes \tau_1) \cong (\text{End}_S(M_2), \tau_2)$ , and  $\sigma_2 = \sigma_{\tau_2}$ . Since  $\sigma_2$  is an induced  $Q$ -normal structure, the homomorphism  $\pi^{\text{Aut}(M_2, Q)}: \text{Aut}(M_2, Q) \rightarrow Q$  is surjective; by Lemma 9.6, the induced homomorphism  $\pi^{\text{Aut}(M_2, Q, \tau_2)}: \text{Aut}(M_2, Q, \tau_2) \rightarrow Q$  is surjective as well whence the  $Q$ -equivariant structure  $\tau_2$  on  $M_2$  is induced.

By Proposition 9.8, with respect to associated group extensions  $e_1: U(S) \xrightarrow{i_1} \Gamma_1 \xrightarrow{\pi_1} Q$  and  $e_2: U(S) \xrightarrow{i_1} \Gamma_2 \xrightarrow{\pi_2} Q$  and morphisms

$$\begin{aligned} (j_1, \beta_1): (U(S), \Gamma_1, i_1) &\longrightarrow (\text{Aut}_S(M_1), \text{Aut}(\text{End}_S(M_1)), \partial_1), \\ (j_2, \beta_2): (U(S), \Gamma_2, i_2) &\longrightarrow (\text{Aut}_S(M_2), \text{Aut}(\text{End}_S(M_2)), \partial_2) \end{aligned}$$

of crossed modules inducing the  $Q$ -equivariant structures  $\tau_1$  and  $\tau_2$ , respectively, the  $Q$ -equivariant  $S$ -algebra  $(\text{End}_S(M_1), \tau_1)$  is equivariantly Brauer equivalent to a  $Q$ -equivariant  $S$ -algebra of the kind  $(\text{End}_S(M_{e_1}), \tau_{e_1})$  and the  $Q$ -equivariant  $S$ -algebra  $(\text{End}_S(M_2), \tau_2)$  to a  $Q$ -equivariant  $S$ -algebra of the kind  $(\text{End}_S(M_{e_2}), \tau_{e_2})$ . Consequently the  $Q$ -equivariant  $S$ -algebra  $(A, \tau)$  is equivariantly Brauer equivalent to a  $Q$ -equivariant  $S$ -algebra of the kind  $(\text{End}_S(M_e), \tau_e)$ , for some group extension  $e: U(S) \twoheadrightarrow \Gamma \twoheadrightarrow Q$  such that

$$[e] + [e_1] = [e_2] \in H^2(Q, U(S))$$

whence  $\text{cpr}([e]) = [(A, \tau)] \in \text{EB}(S, Q)$ .

Suppose now that  $S|R$  is a Galois extension of commutative rings with Galois group  $Q$ . Then  $(A, Q, e, \kappa_Q \circ \pi^\Gamma)$  is an Azumaya  $R$ -algebra, see, e.g., Proposition 4.4(xi) above; moreover, by Proposition 4.4(ix) above, the obvious homomorphism

$$S \otimes_R (A, Q, e, \kappa_Q \circ \pi^\Gamma)^{\text{op}} \longrightarrow \text{End}_S(M_e)$$

is then an isomorphism of  $S$ -algebras and, by Proposition 4.4(x), the  $Q$ -equivariant structure  $\tau_e$  comes from scalar extension; hence the canonical homomorphism  $B(R) \rightarrow \text{EB}(S, Q)$  is then an isomorphism. Likewise, again by Galois descent, the canonical homomorphisms  $H^1(Q, U(S)) \rightarrow \text{Pic}(S|R)$  and  $\text{Pic}(R) \rightarrow \text{EPic}(S, Q)$  are isomorphisms.

We leave the proofs of the naturality assertions to the reader.  $\square$

Given a morphism  $(f, \varphi): (S, Q, \kappa) \rightarrow (T, G, \lambda)$  in the change of actions category *Change* introduced in Subsection 2.7 above, the group  $Q$  being finite, the corresponding relative version of the exact sequence (12.1) takes the following form:

$$\dots \xrightarrow{\omega_{\text{Pic}_S, Q}} H^2(Q, U(S)) \xrightarrow{\text{cpr}} \text{EB}(T|S; G, Q) \xrightarrow{\text{res}} \text{XB}(T|S; G, Q) \xrightarrow{t} H^3(Q, U(S)) \quad (12.5)$$

*Remark 12.3.* In the special case where  $T = S$  and  $G$  is the trivial group, in view of the isomorphism (10.3) from  $\text{XB}(S|S; \{e\}, Q)$  onto  $H^1(Q, \text{Pic}(S))$ , the exact sequence (12.5) has the form of the C(hase-)R(osenberg-)A(uslander-)B(umer) sequence [CR65, Theorem 7.6 p. 62], [Bru66]. Other versions of the CRAB-sequence were obtained by Childs [Chi72, Theorem 2.2], Fröhlich and Wall [FW71a, Theorem 1], [FW71b], [FW00, Theorem 4.2] (upper and middle long sequence), Hattori [Hat79], Kanzaki [Kan68], Ulbrich [Ul79], Yokogawa [Yok78], and Villamayor-Zelinski [VZ78].

Consider a  $Q$ -normal Galois extension  $T|S$  of commutative rings, with structure extension  $e_{(T|S)}: N \rightarrowtail G \xrightarrow{\pi_Q} Q$  and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ , cf. Section 6 above, and denote the injection of  $S$  into  $T$  by  $i: S \rightarrow T$ . Then the abelian groups  $\text{EB}(T|S; G, Q)$  and  $\text{XB}(T|S; G, Q)$  are defined relative to the morphism  $(i, \pi_Q): (S, Q, \kappa_Q) \rightarrow (T, G, \kappa_G)$  in the change of actions category *Change* associated to the data, cf. (6.3) above.

**Theorem 12.4.** *The group  $Q$  being finite, the extension*

$$\begin{aligned} 0 \longrightarrow H^1(Q, U(S)) &\xrightarrow{j_{\text{Pic}_S, Q}} \text{EPic}(S, Q) \xrightarrow{\mu_{\text{Pic}_S, Q}} (\text{Pic}(S))^Q \xrightarrow{\omega_{\text{Pic}_S, Q}} H^2(Q, U(S)) \\ &\xrightarrow{\text{cpr}} \text{EB}(T|S; G, Q) \xrightarrow{\text{res}} \text{XB}(T|S; G, Q) \xrightarrow{t} H^3(Q, U(S)) \xrightarrow{\text{inf}} H^3(G, U(T)) \end{aligned} \quad (12.6)$$

*of the exact sequence (2.15) is defined and yields an eight term exact sequence that is natural in terms of the data. If, furthermore,  $S|R$  and  $T|R$  are Galois extensions of commutative rings over  $R = S^Q = T^G$ , with Galois groups  $Q$  and  $G$ , respectively, then, with  $\text{Pic}(S|R)$ ,  $\text{Pic}(R)$  and  $B(T|R)$  substituted for, respectively  $H^1(Q, U(S))$ ,  $\text{EPic}(S, Q)$  and  $\text{EB}(S, Q)$ , where  $R = S^Q$ , the homomorphisms  $\text{cpr}$  and  $\text{res}$  being modified accordingly, the sequence is exact as well.*

*Proof.* This is an immediate consequence of Theorems (11.2) and (12.1).  $\square$

*Remark 12.5.* A homomorphism of the kind (11.4) above is given in [FW00, Theorem 4.2], with the notation  $B_0(R; \Gamma)$  for what corresponds to our  $\text{EB}(S|S; Q, Q)$  (where our notation  $Q$  and  $S$  corresponds to  $\Gamma$  and  $R$ , respectively). After the statement of Theorem 4.2, the authors of [FW00] remark that there is no direct construction for the map from  $H^2(\Gamma; U(R))$  to  $B_0(R; \Gamma)$ . Our construction of (11.4) is direct, however.



### 13 Relationship with the eight term exact sequence in the cohomology of a group extension

Let  $T|S$  be a  $Q$ -normal Galois extension of commutative rings, with structure extension  $e_{(T|S)}: N \rightarrow G \xrightarrow{\pi_Q} Q$  and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ , cf. Section 6 above; in particular,  $N$  is a finite group. Since  $U(T)^N$  coincides with  $U(S)$ , the eight term exact sequence in [Hue81b] associated to the group extension  $e_{(T|S)}$  and the  $G$ -module  $U(T)$ , reproduced as (7.4) above, has the following form:

$$\begin{aligned} 0 \longrightarrow H^1(Q, U(S)) &\xrightarrow{\text{inf}} H^1(G, U(T)) \xrightarrow{\text{res}} H^1(N, U(T))^Q \xrightarrow{\Delta} H^2(Q, U(S)) \\ &\xrightarrow{\text{inf}} H^2(G, U(T)) \xrightarrow{j} \text{Xpext}(G, N; U(T)) \xrightarrow{\Delta} H^3(Q, U(S)) \xrightarrow{\text{inf}} H^3(G, U(T)). \end{aligned} \quad (13.1)$$

#### 13.1 Relationship between the two long exact sequences

Consider the morphism  $(i, \pi_Q): (S, Q, \kappa_Q) \rightarrow (T, G, \kappa_G)$  associated to the given  $Q$ -normal Galois extension, cf. 6.3, in the change of actions category *Change* introduced in Subsection 2.7 above. The abelian groups  $\text{EB}(T|S; G, Q)$  and  $\text{XB}(T|S; G, Q)$  are now defined relative to this morphism.

The assignment to a crossed pair  $(e: U(T) \rightarrow \Gamma \rightarrow N, \psi: Q \rightarrow \text{Out}_G(e))$  with respect to  $e_{(T|S)}$  and  $U(T)$  of its associated the crossed pair algebra  $(A_e, \sigma_\psi)$ , cf. Section 7 above, yields a homomorphism

$$\text{cpa}: \text{Xpext}(G, N; U(T)) \rightarrow \text{XB}(T|S; G, Q). \quad (13.2)$$

Let  $\text{EPic}(T|S, Q)$  denote the kernel of the induced homomorphisms

$$\text{EPic}(S, Q) \xrightarrow{\mu_{\text{Pic}_{S,Q}}} \text{Pic}(S) \xrightarrow{i_*} \text{Pic}(T)$$

and  $\text{Pic}(T|S)$  that of the induced homomorphism  $i_*: \text{Pic}(S) \rightarrow \text{Pic}(T)$ . With  $T$  and  $G$  substituted for  $Q$  and  $S$ , respectively, the homomorphism (2.17) takes the form

$$j_{\text{Pic}_{T,G}}: H^1(G, U(T)) \rightarrow \text{EPic}(T|T, G), \quad (13.3)$$

and Galois descent yields an isomorphism  $\text{EPic}(T|S, Q) \rightarrow \text{EPic}(T|T, G)$  whence (13.3) induces a homomorphism

$$H^1(G, U(T)) \rightarrow \text{EPic}(T|S, Q) \quad (13.4)$$

of abelian groups. The homomorphism (13.4) admits, of course, a straightforward direct description. Likewise, with  $T$  and  $N$  substituted for  $Q$  and  $S$ , respectively, the isomorphism (2.17) takes the form

$$j_{\text{Pic}_{T,N}}: H^1(N, U(T)) \rightarrow \text{EPic}(T|T, N), \quad (13.5)$$

and Galois descent yields an isomorphism  $\text{Pic}(T|S) \rightarrow \text{EPic}(T|T, N)$  whence (13.5) induces an isomorphism

$$H^1(N, U(T)) \rightarrow \text{Pic}(T|S) \quad (13.6)$$

of abelian groups, necessarily compatible with the  $Q$ -module structures; the isomorphism (13.6) is entirely classical. Below we shall not distinguish in notation between (13.4) and its composite  $H^1(G, U(T)) \rightarrow \text{EPic}(T, Q)$  with the canonical injection of  $\text{EPic}(T|S, Q)$  into  $\text{EPic}(T, Q)$ , nor between (13.6) and its composite  $H^1(N, U(T)) \rightarrow \text{Pic}(S)$  with the canonical injection  $\text{Pic}(T|S) \rightarrow \text{Pic}(S)$ . Direct inspection establishes the following.

**Theorem 13.1.** *The group  $Q$  being finite, the homomorphisms (13.4), (13.6), (11.3), and (13.2) of abelian groups are natural on the category *Change* and induce a morphism of exact sequences from (13.1) to (12.6).  $\square$*

*Remark 13.2.* Consider the classical case where  $R$ ,  $S$ , and  $T$  are fields. Now the group  $\text{Xpext}(G, N; \text{U}(T))$  comes down to  $\text{H}^2(N, \text{U}(T))^Q$  and  $\text{XB}(T|S; G, Q)$  to  $\text{B}(T|S)^Q$ , and (13.2) boils down to the classical isomorphism  $\text{H}^2(N, \text{U}(T))^Q \rightarrow \text{B}(T|S)^Q$ . Furthermore, the groups  $\text{H}^1(N, \text{U}(T))$ ,  $\text{H}^1(G, \text{U}(T))$ ,  $\text{EPic}(T|S, Q)$ , and  $\text{Pic}(T|S)$  are zero, and (11.3) is an isomorphism. Thus the morphism (13.1)  $\rightarrow$  (12.6) of exact sequences in Theorem 13.1 above is then an isomorphism of exact sequences.

### 13.2 An application

Let  $T|S$  be a Galois extension of commutative rings, with Galois group  $N$ , suppose that  $T$  carries a  $Q$ -action that extends the given  $Q$ -action on  $S$ , and define the group  $\text{EB}(T|S, Q)$  to be the kernel of the induced homomorphism  $\text{EB}(S, Q) \rightarrow \text{EB}(T, Q)$ . Relative to the induced  $Q$ -action on  $N$ , the semi-direct product group  $N \rtimes Q$  is defined, and  $T|S$  is a  $Q$ -normal Galois extension of rings, having as structure extension the split extension  $e_{(T|S)}: N \rightarrow N \rtimes Q \rightarrow Q$ . Consider the commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \widetilde{\text{EB}(T|S, Q)} & \longrightarrow & \text{EB}(T|S; N \rtimes Q, Q) & \longrightarrow & \text{EB}(T|T; Q, Q) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{EB}(T|S, Q) & \longrightarrow & \text{EB}(S, Q) & \longrightarrow & \text{EB}(T, Q) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{XB}(T; Q, N \rtimes Q) & \longrightarrow & \text{XB}(T, N \rtimes Q) & \longrightarrow & \text{XB}(T, Q)
\end{array} \tag{13.7}$$

of abelian groups with exact rows and columns, the subgroup  $\widetilde{\text{EB}(T|S, Q)}$  of  $\text{EB}(T|S, Q)$  being defined by the requirement that the upper row be exact.

The group  $N$  being finite, suppose now that  $Q$  is a finite group as well. The corresponding homomorphism (11.3), viz.  $\text{cpr}: \text{H}^2(N \rtimes Q, \text{U}(T)) \rightarrow \text{EB}(T|S; N \rtimes Q, Q)$ , and the homomorphism (11.4), with  $T$  substituted for  $S$ , viz.  $\text{cpr}: \text{H}^2(Q, \text{U}(T)) \rightarrow \text{EB}(T|T; Q, Q)$ , yield the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker(\text{res}) & \longrightarrow & \text{H}^2(N \rtimes Q, \text{U}(T)) & \xrightarrow{\text{res}} & \text{H}^2(Q, \text{U}(T)) \\
& & \downarrow & & \text{cpr} \downarrow & & \text{cpr} \downarrow \\
0 & \longrightarrow & \widetilde{\text{EB}(T|S, Q)} & \longrightarrow & \text{EB}(T|S; N \rtimes Q, Q) & \longrightarrow & \text{EB}(T|T; Q, Q)
\end{array} \tag{13.8}$$

with exact rows and hence a homomorphism  $\ker(\text{res}) \rightarrow \widetilde{\text{EB}(T|S, Q)}$  of abelian groups. Suppose, furthermore, that  $S$  and  $T$  are fields. Then the homomorphism from  $\text{XB}(T, N \rtimes Q)$  to  $\text{XB}(T, Q)$  in the lower row of the diagram (13.7) comes down to the obvious injection  $\text{B}(T)^{N \rtimes Q} \rightarrow \text{B}(T)^Q$  whence the group  $\text{XB}(T; Q, N \rtimes Q)$  is now trivial and the inclusion  $\widetilde{\text{EB}(T|S, Q)} \subseteq \text{EB}(T|S, Q)$  is the identity. Moreover, the right-hand and the middle vertical

arrow in (13.8) are isomorphisms whence the induced homomorphism  $\ker(\text{res}) \rightarrow \text{EB}(T|S, Q)$  is an isomorphism. This observation recovers and casts new light on the main result of [CG06], obtained there via relative group cohomology. Our argument is elementary and does not invoke relative group cohomology. Indeed, the main point of our reasoning is the identification of the group cohomology group  $H^2(N \rtimes Q, U(T))$  with the group  $\text{EB}(T|S; N \rtimes Q, Q)$ ; under the present circumstances, this group is the subgroup of the  $Q$ -equivariant Brauer group  $\text{EB}(S, Q)$  of  $S$  that consists of classes of  $Q$ -equivariant central simple  $S$ -algebras  $A$  such that  $A \otimes T$  is a matrix algebra over  $T$ . Likewise, the group  $\text{EB}(T|T; Q, Q)$  is the subgroup of the  $Q$ -equivariant Brauer group  $\text{EB}(T, Q)$  of  $T$  that consists of classes of  $Q$ -equivariant matrix algebras over  $T$ . The group  $\text{EB}(T|S, Q)$  then appears as the kernel of the canonical homomorphism  $\text{EB}(T|S; N \rtimes Q, Q) \rightarrow \text{EB}(T|T; Q, Q)$  and, in view of the identifications of  $H^2(N \rtimes Q, U(T))$  with  $\text{EB}(T|S; N \rtimes Q, Q)$  and of  $H^2(Q, U(T))$  with  $\text{EB}(T|T; Q, Q)$ , the identification of  $\ker(\text{res}: H^2(N \rtimes Q, U(T)) \rightarrow H^2(Q, U(T)))$  with  $\text{EB}(T|S, Q)$  is immediate. In particular, when the group  $Q$  is trivial, that result comes down to the classical Brauer-Hasse-Noether isomorphism between the corresponding second group cohomology group and the corresponding subgroup of the ordinary Brauer group.

### 13.3 A variant of the relative theory

In the situation of the relative versions (12.5) and (12.6) of the long exact sequence (12.1), there is no obvious reason for a homomorphism  $\omega$  from  $H^0(Q, B(T|S))$  to  $H^2(Q, \text{Pic}(T|S))$  to exist in general that would complete

$$\begin{array}{ccc} H^0(Q, B(T|S)) & & H^2(Q, \text{Pic}(T|S)) \\ \downarrow & & \downarrow \\ H^0(Q, B(S)) & \xrightarrow{\omega} & H^2(Q, \text{Pic}(S)) \end{array}$$

to a commutative square and hence would complete the exact sequence (10.7) to a corresponding relative version of an exact sequence of the kind (2.14) above. We shall now show that a variant of the relative theory includes such a homomorphism.

The object  $(S, Q, \kappa_Q)$  of the category *Change* being given, let  $(T, G, \kappa_G)$  be another object of *Change*, and let  $(f, \varphi): (S, Q, \kappa_Q) \rightarrow (T, G, \kappa_G)$  be a morphism in *Change* having  $\varphi: G \rightarrow Q$  surjective, cf. Subsection 2.7 above.

#### 13.3.1 The standard approach

We will say that two  $Q$ -normal Azumaya  $S$ -algebras  $(A_1, \sigma_1)$  and  $(A_2, \sigma_2)$  such that  $T \otimes A_1$  and  $T \otimes A_2$  are matrix algebras over  $T$  are *relatively Brauer equivalent* if there are faithful finitely generated projective  $S$ -modules  $M_1$  and  $M_2$  having the property that  $T \otimes M_1$  and  $T \otimes M_2$  are free as  $T$ -modules, together with induced  $Q$ -normal structures

$$\rho_1: Q \rightarrow \text{Out}(B_1), \quad B_1 = \text{End}_S(M_1), \quad \rho_2: Q \rightarrow \text{Out}(B_2), \quad B_2 = \text{End}_S(M_2),$$

such that  $(A_1 \otimes B_1, \sigma_1 \otimes \rho_1)$  and  $(A_2 \otimes B_2, \sigma_2 \otimes \rho_2)$  are isomorphic  $Q$ -normal  $S$ -algebras. Just as for  $\text{XB}(S, Q)$ , under the operations of tensor product and that of taking opposite algebras, the equivalence classes constitute an abelian group, the identity element being represented by  $(S, \kappa_Q)$ . We refer to this group as the  *$T$ -relative  $Q$ -crossed Brauer group of  $S$  with respect*

to the morphism  $(f, \varphi)$  in *Change*, denote this group by  $\text{XB}_{\text{fr}}(T|S; G, Q)$ , and we refer to the construction just given as the *standard construction*. The  $T$ -relative  $Q$ -equivariant Brauer group  $\text{EB}_{\text{fr}}(T|S; G, Q)$  with respect to the morphism  $(f, \varphi)$  in *Change* arises in the same way as the relative  $Q$ -crossed Brauer group, save that, in the definition, ‘equivariant’ is substituted for ‘crossed’, and we will likewise say that this construction is the *standard construction*. In particular, when we forget the actions, that is, we take the groups  $G$  and  $Q$  to be trivial, this construction yields an abelian group  $\text{B}_{\text{fr}}(T|S)$  which we refer to as the  $T$ -relative Brauer group of  $S$ , obtained by the *standard construction*.

The group  $\text{B}_{\text{fr}}(T|S)$  acquires a  $Q$ -module structure. Indeed, let  $R = S^Q$ . Given an  $S$ -module  $M$  and  $x \in Q$ , let  ${}^x M$  denote the  $S$ -module whose underlying  $R$ -module is just  $M$ , and whose  $S$ -module structure is given by

$$S \otimes M \longrightarrow M, (s \otimes q) \longmapsto {}^x s q, s \in S, q \in M.$$

Consider a faithful finitely generated projective  $S$ -module  $M$  such that  $T \otimes M$  is a free  $T$ -module, let  $x \in Q$ , and pick a pre-image  $y \in G$  of  $x \in Q$ . Then the association

$$T \otimes {}^x M \longrightarrow {}^y(T \otimes M), t \otimes q \longmapsto {}^y t \otimes q, \quad (13.9)$$

yields an isomorphism of  $T$ -modules, and since  $T \otimes M$  is a free  $T$ -module, so is  ${}^y(T \otimes M)$ ; further,

$$T \otimes {}^x \text{End}_S(M) \cong {}^y(T \otimes \text{End}_S(M)) \cong {}^y(T \otimes \text{End}_S(M)) \cong \text{End}_S({}^y(T \otimes M))$$

is a matrix algebra over  $T$ . Likewise, given an Azumaya  $S$ -algebra  $A$  such that  $T \otimes A$  is a matrix algebra over  $T$  and  $x \in Q$ , to show that  $T \otimes {}^x A$  is a matrix algebra over  $T$ , pick a pre-image  $y \in G$  of  $x \in Q$  and note that the corresponding association (13.9) yields an isomorphism of  $T$ -algebras. Since  $T \otimes A$  is a matrix algebra over  $T$ , so is  ${}^y(T \otimes A)$ .

By construction, the canonical homomorphism

$$\text{B}_{\text{fr}}(T|S) \longrightarrow \text{B}(T|S) \quad (13.10)$$

is a morphism of  $Q$ -modules but in general there is no reason for this homomorphism to be injective nor to be surjective. The assignment to a  $Q$ -equivariant Azumaya  $S$ -algebra representing a member of  $\text{EB}_{\text{fr}}(T|S; G, Q)$  of the associated  $Q$ -normal Azumaya  $S$ -algebra yields a homomorphism  $\text{res}_{\text{fr}}: \text{EB}_{\text{fr}}(T|S; G, Q) \rightarrow \text{XB}_{\text{fr}}(T|S; G, Q)$  of abelian groups, the assignment to a  $Q$ -normal Azumaya  $S$ -algebra  $(A, \sigma)$  representing a member of  $\text{XB}_{\text{fr}}(T|S; G, Q)$  of its Teichmüller complex  $e_{(A, \sigma)}$  yields a homomorphism  $t_{\text{fr}}: \text{XB}_{\text{fr}}(T|S; G, Q) \longrightarrow \text{H}^3(Q, \text{U}(S))$  of abelian groups and, when the group  $Q$  is finite, the construction of the homomorphism  $\text{cpr}: \text{H}^2(Q, \text{U}(S)) \rightarrow \text{EB}(T|S; G, Q)$ , cf. (11.4) above, lifts to a homomorphism

$$\text{cpr}_{\text{fr}}: \text{H}^2(Q, \text{U}(S)) \longrightarrow \text{EB}_{\text{fr}}(T|S; G, Q).$$

*Remark 13.3.* The abelian groups  $\text{EB}(T|S; G, Q)$  and  $\text{XB}(T|S; G, Q)$  being defined relative to the given morphism  $(f, \varphi)$  in *Change*, the obvious maps yields homomorphisms

$$\text{EB}_{\text{fr}}(T|S; G, Q) \longrightarrow \text{EB}(T|S; G, Q) \quad (13.11)$$

$$\text{XB}_{\text{fr}}(T|S; G, Q) \longrightarrow \text{XB}(T|S; G, Q) \quad (13.12)$$

of abelian groups that make the diagram

$$\begin{array}{ccccc}
\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) & \xrightarrow{\mathrm{res}_{\mathrm{fr}}} & \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q) & \xrightarrow{t_{\mathrm{fr}}} & \mathrm{H}^3(Q, \mathrm{U}(S)) \\
\downarrow & & \downarrow & & \parallel \\
\mathrm{EB}(T|S; G, Q) & \xrightarrow{\mathrm{res}} & \mathrm{XB}(T|S; G, Q) & \xrightarrow{t} & \mathrm{H}^3(Q, \mathrm{U}(S))
\end{array}$$

commutative and, when the group  $Q$  is finite, the homomorphisms  $\mathrm{cpr}_{\mathrm{fr}}$  from  $\mathrm{H}^2(Q, \mathrm{U}(S))$  to  $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$  and  $\mathrm{cpr}$  from  $\mathrm{H}^2(Q, \mathrm{U}(S))$  to  $\mathrm{EB}(T|S; G, Q)$  extend the diagram to a larger commutative diagram having four terms in each row. However, there is no reason for the homomorphisms (13.11) or (13.12) to be injective nor to be surjective, nor is there a reason, when  $Q$  is a finite group, for  $\mathrm{cpr}_{\mathrm{fr}}: \mathrm{H}^2(Q, \mathrm{U}(S)) \rightarrow \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$  to be injective or surjective. In the classical situation where  $R, S, T$  are fields etc., these homomorphisms are, of course, isomorphisms.

Let  $\mathrm{Pic}(T|S)$  denote the kernel of the homomorphism  $\mathrm{Pic}(S) \rightarrow \mathrm{Pic}(T)$  induced by the ring homomorphism  $f: S \rightarrow T$ , necessarily a morphism of  $G$ -modules when  $G$  acts on  $S$  through  $\varphi: G \rightarrow Q$  whence, in particular, the abelian subgroup  $\mathrm{Pic}(T|S)^Q$  of  $Q$ -invariants is defined, and let  $\mathrm{EPic}(T|S, Q)$  denote the kernel of the homomorphism  $\mathrm{EPic}(S, Q) \rightarrow \mathrm{EPic}(T, G)$  induced by the morphism  $(f, \varphi)$  in *Change*. It is immediate that the low degree exact sequence (2.14) restricts to the exact sequence

$$0 \longrightarrow \mathrm{H}^1(Q, \mathrm{U}(S)) \xrightarrow{j_{\mathrm{Pic}_S, Q}} \mathrm{EPic}(T|S, Q) \xrightarrow{\mu_{\mathrm{Pic}_S, Q}^{\mid}} \mathrm{Pic}(T|S)^Q \xrightarrow{\omega_{\mathrm{Pic}_S, Q}^{\mid}} \mathrm{H}^2(Q, \mathrm{U}(S)) \quad (13.13)$$

of abelian groups. In the appendix (cf. Subsection 15.2 below), we shall show that, with a suitably defined Picard category  $\mathcal{P}\mathrm{ic}_{T|S; G, Q}$  substituted for  $\mathcal{C}_Q$ , the sequence (13.13) is as well a special case of the exact sequence (2.10).

**Theorem 13.4.** *Suppose that the group  $Q$  is finite. Then the extension*

$$\dots \xrightarrow{\omega_{\mathrm{Pic}_S, Q}} \mathrm{H}^2(Q, \mathrm{U}(S)) \xrightarrow{\mathrm{cpr}_{\mathrm{fr}}} \mathrm{EB}_{\mathrm{fr}}(T|S; G, Q) \xrightarrow{\mathrm{res}_{\mathrm{fr}}} \mathrm{XB}_{\mathrm{fr}}(T|S; G, Q) \xrightarrow{t_{\mathrm{fr}}} \mathrm{H}^3(Q, \mathrm{U}(S)) \quad (13.14)$$

*of the exact sequence (13.13) is defined and yields a seven term exact sequence that is natural in terms of the data.*

*Proof.* Essentially the same reasoning as that for Theorem 12.1 establishes this theorem as well. We explain only the requisite salient modifications.

*Exactness at  $\mathrm{XB}_{\mathrm{fr}}(T|S; G, Q)$ :* This follows again from Theorem 5.1 or Theorem 11.1.

*Exactness at  $\mathrm{H}^2(Q, \mathrm{U}(S))$ :* Let  $J$  represent a class in  $(\mathrm{Pic}(T|S))^Q$ , and proceed as in the proof of the exactness at  $\mathrm{H}^2(Q, \mathrm{U}(S))$  in Theorem 12.1. Now  $T \otimes J$  is free as a  $T$ -module and, with reference to the associated group extension  $e_J$ , cf. (12.2), by construction,  $M_{e_J}$  is free as an  $S$ -module whence  $T \otimes M_{e_J}$  is free as a  $T$ -module. Hence

$$T \otimes \mathrm{Hom}_S(J, M_{e_J}) \cong \mathrm{Hom}_T(T \otimes J, T \otimes M_{e_J})$$

is free as a  $T$ -module. Consequently  $(\mathrm{End}_S(M_{e_J}), \tau_{e_J})$  represents zero in  $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$ .

Conversely, let  $e: \mathrm{U}(S) \rightarrow \Gamma \rightarrow Q$  be a group extension, and proceed as in the proof of the exactness at  $\mathrm{H}^2(Q, \mathrm{U}(S))$  in Theorem 12.1. Thus suppose that  $(\mathrm{End}_S(M_e), \tau_e)$  represents zero in  $\mathrm{EB}_{\mathrm{fr}}(T|S; G, Q)$ . Then there are faithful finitely generated projective  $S$ -modules

$M_1$  and  $M_2$  which admit, furthermore,  $S^tQ$ -module structures so that, with the notation  $\tau_1: Q \rightarrow \text{Aut}(\text{End}_S(M_1))$  and  $\tau_2: Q \rightarrow \text{Aut}(\text{End}_S(M_2))$  for the associated trivially induced  $Q$ -equivariant structures, beyond  $(\text{End}_S(M_e), \tau_e) \otimes (\text{End}_S(M_1), \tau_1)$  and  $(\text{End}_S(M_2), \tau_2)$  being isomorphic as  $Q$ -equivariant  $S$ -algebras, the  $T$ -modules  $T \otimes M_1$  and  $T \otimes M_2$  are free as  $T$ -modules. Then the  $S$ -module  $J = \text{Hom}_{\text{End}_S(M_e \otimes M_1)}(M_e \otimes M_1, M_2)$ , beyond being finitely generated and projective of rank one is, furthermore, as a  $T$ -module, free of rank one, whence  $[J] \in \text{Pic}(T|S)$ . The group extension  $e_J$ , cf. (12.2), is now defined relative to  $J$ , whence  $[J] \in (\text{Pic}(T|S))^Q$ , and the  $\Gamma$ -action on  $J$  induces a homomorphism  $\Gamma \rightarrow \text{Aut}(J, Q)$  which yields a congruence  $(1, \cdot, 1): e \rightarrow e_J$  of group extensions, and this congruence entails that  $\omega_{\text{Pic}_{S,Q}}[J] = [e] \in H^2(Q, U(S))$ .

*Exactness at  $\text{EB}_{\text{fr}}(T|S; G, Q)$ :* The reasoning in the proof of Theorem 12.1 which shows that the composite  $\text{res} \circ \text{cpr}$  is zero shows as well that the composite  $\text{res}_{\text{fr}} \circ \text{cpr}_{\text{fr}}$  is zero.

To show that  $\ker(\text{res}_{\text{fr}}) \subset \text{im}(\text{cpr}_{\text{fr}})$ , let  $(A, \tau)$  be a  $Q$ -equivariant Azumaya  $S$ -algebra representing a member of  $\text{EB}_{\text{fr}}(T|S; G, Q)$ , and suppose that the class of its associated  $Q$ -normal algebra  $(A, \sigma_\tau)$  goes to zero in  $\text{XB}_{\text{fr}}(T|S; G, Q)$ . As in the proof of the exactness at  $\text{EB}(S, Q)$  in Theorem 12.1, there are two induced  $Q$ -equivariant split algebras  $(\text{End}_S(M_1), \tau_1)$  and  $(\text{End}_S(M_2), \tau_2)$  over faithful finitely generated projective  $S$ -modules  $M_1$  and  $M_2$ , respectively, such that  $(A, \tau) \otimes (\text{End}_S(M_1), \tau_1)$  and  $(\text{End}_S(M_2), \tau_2)$  are isomorphic as  $Q$ -equivariant central  $S$ -algebras but now we may furthermore take  $M_1$  and  $M_2$  to have the property that the  $T$ -modules  $T \otimes M_1$  and  $T \otimes M_2$  are free of finite rank. Essentially the same reasoning as that in the proof of the exactness at  $\text{EB}(S, Q)$  in Theorem 12.1 establishes a group extension  $e: U(S) \twoheadrightarrow \Gamma \twoheadrightarrow Q$  such that

$$\text{cpr}_{\text{fr}}([e]) = [(\text{End}_S(M_e), \tau_e)] = [(A, \tau)] \in \text{EB}_{\text{fr}}(T|S; G, Q).$$

□

Consider a  $Q$ -normal Galois extension  $T|S$  of commutative rings, with structure extension  $e_{(T|S)}: N \twoheadrightarrow G \xrightarrow{\pi_Q} Q$  and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ , cf. Section 6 above, and take the morphism  $(f, \varphi)$  to be the morphism  $(i, \pi_Q): (S, Q, \kappa_Q) \rightarrow (T, G, \kappa_G)$  in *Change* associated to that  $Q$ -normal Galois extension, cf. (6.3).

**Theorem 13.5.** *Suppose that the group  $Q$  is finite. Then the extension*

$$\begin{aligned} 0 \longrightarrow H^1(Q, U(S)) &\xrightarrow{j_{\text{Pic}_{S,Q}}|} \text{EPic}(T|S, Q) \xrightarrow{\mu_{\text{Pic}_{S,Q}}|} (\text{Pic}(T|S))^Q \xrightarrow{\omega_{\text{Pic}_{S,Q}}|} H^2(Q, U(S)) \\ &\xrightarrow{\text{cpr}_{\text{fr}}|} \text{EB}_{\text{fr}}(T|S; G, Q) \xrightarrow{\text{res}_{\text{fr}}} \text{XB}_{\text{fr}}(T|S; G, Q) \xrightarrow{t_{\text{fr}}} H^3(Q, U(S)) \xrightarrow{\text{inf}} H^3(G, U(T)) \end{aligned} \quad (13.15)$$

*of the exact sequence (13.13) is defined and yields an eight term exact sequence that is natural in terms of the data.*

*Proof.* Essentially the same reasoning as that for Theorem 12.4 establishes this theorem as well. We leave the details to the reader. □

The homomorphism (13.2) now lifts to a homomorphism

$$\text{Xpext}(G, N; U(T)) \longrightarrow \text{XB}_{\text{fr}}(T|S; G, Q) \quad (13.16)$$

such that (13.2) may be written as the composite

$$\text{Xpext}(G, N; U(T)) \longrightarrow \text{XB}_{\text{fr}}(T|S; G, Q) \longrightarrow \text{XB}(T|S; G, Q) \quad (13.17)$$

and, when  $Q$  and hence  $G$  is a finite group, the homomorphism (11.3) lifts to a homomorphism

$$H^2(G, U(T)) \longrightarrow \text{EB}_{\text{fr}}(T|S; G, Q) \quad (13.18)$$

such that (11.3) may be written as the composite

$$H^2(G, U(T)) \longrightarrow \text{EB}_{\text{fr}}(T|S; G, Q) \longrightarrow \text{EB}(T|S; G, Q). \quad (13.19)$$

Theorem 13.1, adjusted to the present circumstances, takes the following form which, again, we spell out without proof.

**Theorem 13.6.** *The group  $Q$  being finite, the maps (13.16), (13.4), (13.6), and (13.18) are natural homomorphisms of abelian groups and induce a morphism  $(13.1) \rightarrow (13.15)$  of exact sequences.*

### 13.3.2 The Morita equivalence approach

We define the  $Q$ -graded relative Brauer precategory associated to the morphism  $(f, \varphi)$  in *Change* to be the precategory  $\text{Pre}\mathcal{B}_{T|S;G,Q}$  that has as its objects the Azumaya  $S$ -algebras  $A$  such that  $T \otimes A$  is a matrix algebra over  $T$ , a morphism  $([M], x): A \rightarrow B$  in  $\text{Pre}\mathcal{B}_{T|S;G,Q}$  of grade  $x \in Q$  between two Azumaya algebras  $A$  and  $B$  in  $\mathcal{B}_{T|S;G,Q}$ , necessarily an isomorphism in  $\text{Pre}\mathcal{B}_{T|S;G,Q}$ , being a morphism in  $\mathcal{B}_{S,Q}$ , that is, a pair  $([M], x)$  where  $[M]$  is an isomorphism class of an invertible  $(B, A)$ -bimodule  $M$  of grade  $x \in Q$ , such that, furthermore,  $T \otimes M$  is free as a  $T$ -module. There is no reason for composition in the ambient category  $\mathcal{B}_{S,Q}$  to induce an operation of composition in  $\text{Pre}\mathcal{B}_{T|S;G,Q}$  since, given three Azumaya algebras  $A, B, C$  in  $\text{Pre}\mathcal{B}_{T|S;G,Q}$  and morphisms  $([{}_B M_A], x): A \rightarrow B$  and  $([{}_A M_C], x): C \rightarrow A$  of grade  $x \in Q$  in  $\text{Pre}\mathcal{B}_{T|S;G,Q}$ , while the composition  $([{}_B M_A \otimes_A {}_A M_C], x): C \rightarrow B$  of grade  $x \in Q$  in  $\mathcal{B}_{S,Q}$  is defined, there is no reason for the  $(T \otimes B, T \otimes C)$ -bimodule

$$T \otimes ({}_B M_A \otimes_A {}_A M_C) \cong {}_{T \otimes B}(T \otimes M)_{T \otimes A} \otimes_{(T \otimes A)} {}_{T \otimes A}(T \otimes M)_{T \otimes C}$$

to be free as a  $T$ -module. To overcome this difficulty, we take the  $Q$ -graded relative Brauer category associated to the morphism  $(f, \varphi)$  in *Change* to be the subcategory  $\mathcal{B}_{T|S;G,Q}$  of  $\mathcal{B}_{S,Q}$  generated by  $\text{Pre}\mathcal{B}_{T|S;G,Q}$ . Thus a morphism in  $\mathcal{B}_{T|S;G,Q}$  of grade  $x \in Q$  between two objects  $A$  and  $B$  of  $\mathcal{B}_{T|S;G,Q}$  is a morphism  $([{}_B M_A], x): A \rightarrow B$  in  $\mathcal{B}_{S,Q}$  of grade  $x \in Q$  such that there are objects  $A_1, \dots, A_n$  of  $\mathcal{B}_{T|S;G,Q}$  and morphisms  $([{}_{A_{j+1}} M_{A_j}], x): A_j \rightarrow A_{j+1}$  in  $\text{Pre}\mathcal{B}_{T|S;G,Q}$  such that, with the notation  $A_0 = A$  and  $A_n = B$ ,

$${}_B M_A \cong {}_{A_n} M_{A_{n-1}} \otimes_{A_{n-1}} \dots \otimes_{A_2} {}_{A_2} M_{A_1} \otimes_{A_1} {}_{A_1} M_{A_0}. \quad (13.20)$$

We then define composition, monoidal structure, the operation of inverse, and the unit object as in  $\mathcal{B}_{S,Q}$ . The resulting category  $\mathcal{B}_{T|S;G,Q}$  is a group-like stably  $Q$ -graded symmetric monoidal category. Hence the category  $\text{Rep}(Q, \mathcal{B}_{T|S;G,Q})$  is group-like and thence  $k\text{Rep}(Q, \mathcal{B}_{T|S;G,Q})$  is an abelian group. When the groups  $G$  and  $Q$  are trivial, that is, we consider merely the homomorphism  $f: S \rightarrow T$  of commutative rings, the same construction yields a precategory  $\text{Pre}\mathcal{B}_{T|S}$  and, accordingly, the corresponding group-like symmetric monoidal category  $\mathcal{B}_{T|S}$  which we refer to as the *relative Brauer category associated to the homomorphism  $f: S \rightarrow T$*  of commutative rings. The category  $\mathcal{B}_{T|S}$  has  $U(\mathcal{B}_{T|S}) = \text{Pic}(T|S)$  as its unit group, is group-like, and  $k\mathcal{B}_{T|S}$  is therefore an abelian group. The ring homomorphism  $f: S \rightarrow T$  being a constituent of the morphism  $(f, \varphi)$  in *Change* having  $\varphi$  surjective,

the category  $\mathcal{B}_{T|S;G,Q}$  has  $\mathcal{Ker}(\mathcal{B}_{T|S;G,Q}) = \mathcal{B}_{T|S}$  and  $U(\mathcal{B}_{T|S;G,Q}) = U(\mathcal{B}_{T|S}) = \text{Pic}(T|S)$  as its unit group.

Given two objects  $A$  and  $B$  of  $\mathcal{B}_{T|S;G,Q}$  we define, with respect to the morphism  $(f, \varphi)$  in *Change*, a relative Morita equivalence of grade  $x \in Q$  between  $A$  and  $B$  to be a string of isomorphisms in  $\text{Pre}\mathcal{B}_{T|S;G,Q}$  of the kind (13.20) above. It is immediate that, as in the classical situation, given two objects  $A_1$  and  $A_2$  of  $\mathcal{B}_{T|S}$ , a relative Brauer equivalence  $A_1 \otimes \text{End}_S(M_1) \cong A_2 \otimes \text{End}_S(M_2)$  between  $A_1$  and  $A_2$  induces a string

$$A_1 \simeq A_1 \otimes \text{End}_S(M_1) \cong A_2 \otimes \text{End}_S(M_2) \simeq A_2$$

of isomorphisms in  $\text{Pre}\mathcal{B}_{T|S}$  and hence a relative Morita equivalence between  $A_1$  and  $A_2$  (of grade  $e \in Q$ ) whence the obvious association induces a homomorphism

$$\text{B}_{\text{fr}}(T|S) \longrightarrow k\mathcal{B}_{T|S} \quad (13.21)$$

of abelian groups, necessarily surjective. Moreover, since  $\mathcal{Ker}(\mathcal{B}_{T|S;G,Q})$  is stably graded,  $k\mathcal{B}_{T|S} = k\mathcal{Ker}(\mathcal{B}_{T|S;G,Q})$  acquires a  $Q$ -module structure, and the homomorphism (13.21) is a morphism of  $Q$ -modules.

**Proposition 13.7.** *The homomorphism (13.21) is an isomorphism, that is, relative Brauer equivalence is equivalent to relative Morita equivalence.*

*Proof.* The classical argument, suitably rephrased, carries over: Let  $A$  and  $B$  be two Azumaya  $S$ -algebras  $A$  in  $\mathcal{B}_{T|S}$  and consider a morphism  $[M]: A \rightarrow B$  in  $\text{Pre}\mathcal{B}_{T|S}$ . We must show that  $A$  and  $B$  are relatively Brauer equivalent. Now  $B^{\text{op}} \cong {}_A \text{End}(M)$  (the algebra of left  $A$ -endomorphisms of  $M$ ), and

$$\text{End}_S(M) \cong A \otimes ({}_A \text{End}(M)) \cong A \otimes B^{\text{op}}$$

whence

$$\text{End}_S(M) \otimes B \cong A \otimes B^{\text{op}} \otimes B \cong A \otimes \text{End}_S(B).$$

Since  $T \otimes M$  and  $T \otimes B$  are free as  $T$ -modules,  $A$  and  $B$  are relatively Brauer equivalent.  $\square$

With  $N$ ,  $T$ ,  $S$  substituted for, respectively,  $Q$ ,  $S$ ,  $R$ , the standard homomorphism (4.6) from  $H^2(N, U)$  to  $\text{B}(T|S)$ , necessarily a morphism of  $Q$ -modules, lifts to a morphism

$$H^2(N, U) \longrightarrow \text{B}_{\text{fr}}(T|S) \quad (13.22)$$

of  $Q$ -modules. By construction, then, the assignment to an automorphism in  $\mathcal{B}_{T|S;G,Q}$  of an Azumaya algebra  $A$  in  $\mathcal{B}_{T|S;G,Q}$  of its grade in  $Q$  yields a homomorphism

$$\pi^{\text{Aut}_{\mathcal{B}_{T|S;G,Q}}(A)}: \text{Aut}_{\mathcal{B}_{T|S;G,Q}}(A) \longrightarrow Q \quad (13.23)$$

which is surjective if and only if the Brauer class  $[A] \in \text{B}_{\text{fr}}(T|S)$  of  $A$  in  $\text{B}_{\text{fr}}(T|S) \cong k\mathcal{B}_{T|S}$  is fixed under  $Q$ , and the group  $\text{Aut}_{\mathcal{B}_{T|S;G,Q}}(A)$  associated to an Azumaya  $S$ -algebra  $A$  in  $\mathcal{B}_{T|S}$  whose Brauer class  $[A] \in \text{B}_{\text{fr}}(T|S)$  is fixed under  $Q$  fits into a group extension of the kind (2.6), viz.

$$e_A^{\text{Pic}(T|S)}: 1 \longrightarrow \text{Pic}(T|S) \longrightarrow \text{Aut}_{\mathcal{B}_{T|S;G,Q}}(A) \xrightarrow{\pi^{\text{Aut}_{\mathcal{B}_{T|S;G,Q}}(A)}} Q \longrightarrow 1 \quad (13.24)$$



with abelian kernel in such a way that the assignment to  $A$  of  $e_A^{\text{Pic}(T|S)}$  yields a homomorphism

$$\omega_{\mathcal{B}_{T|S;G,Q}}: H^0(Q, B_{\text{fr}}(T|S)) \longrightarrow H^2(Q, \text{Pic}(T|S)). \quad (13.25)$$

The sequence (2.10) now takes the form

$$0 \longrightarrow H^1(Q, \text{Pic}(T|S)) \xrightarrow{j_{\mathcal{B}_{T|S;G,Q}}} k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q}) \xrightarrow{\mu_{\mathcal{B}_{T|S;G,Q}}} B_{\text{fr}}(T|S)^Q \xrightarrow{\omega_{\mathcal{B}_{T|S;G,Q}}} H^2(Q, \text{Pic}(T|S)) \quad (13.26)$$

and is an exact sequence of abelian groups since the category  $\mathcal{B}_{T|S;G,Q}$  is group-like. Furthermore, the association that defines the homomorphism (10.4) above yields an injective homomorphism

$$\theta_{\text{fr}}: \text{XB}_{\text{fr}}(T|S; G, Q) \longrightarrow k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q}) \quad (13.27)$$

in such a way that the diagram

$$\begin{array}{ccc} \text{XB}_{\text{fr}}(T|S; G, Q) & \xrightarrow{\theta_{\text{fr}}} & k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q}) \\ \downarrow & & \downarrow \\ \text{XB}(S, Q) & \xrightarrow{\theta} & k\mathcal{R}ep(Q, \mathcal{B}_{S,Q}) \end{array}$$

is commutative, the unlabeled vertical arrows being the obvious maps, and the argument for Theorem 10.10 (iii), adjusted to the present situation, shows that if  $Q$  (and hence  $G$ ) is a finite group, the homomorphism  $\theta_{\text{fr}}$  is surjective and hence an isomorphism of abelian groups. Thus when the group  $Q$  is finite, the exact sequence (13.26) is available with  $\text{XB}_{\text{fr}}(T|S; G, Q)$  substituted for  $k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q})$ .

Consider a  $Q$ -normal Galois extension  $T|S$  of commutative rings, with structure extension  $e_{(T|S)}: N \rightarrowtail G \xrightarrow{\pi_Q} Q$  and structure homomorphism  $\kappa_G: G \rightarrow \text{Aut}^S(T)$ , cf. Section 6 above, and take the morphism  $(f, \varphi)$  to be the morphism  $(i, \pi_Q): (S, Q, \kappa_Q) \longrightarrow (T, G, \kappa_G)$  in *Change* associated to that  $Q$ -normal Galois extension, cf. 6.3. Comparison of the exact sequences (2.14) and (13.26) with [Hue81b, (1.9)] yields the following result, which we spell out without proof.

**Theorem 13.8.** *Write  $U = U(T)$ . The various groups and homomorphisms fit into a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(Q, H^1(N, U)) & \longrightarrow & \text{Xpext}(G, N; U) & \longrightarrow & H^0(Q, H^2(N, U)) \xrightarrow{d_2} H^2(Q, H^1(N, U)) \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & H^1(Q, \text{Pic}(T|S)) & \xrightarrow{j} & k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q}) & \xrightarrow{\mu} & H^0(Q, B_{\text{fr}}(T|S)) \xrightarrow{\omega} H^2(Q, \text{Pic}(T|S)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(Q, \text{Pic}(S)) & \xrightarrow{j} & k\mathcal{R}ep(Q, \mathcal{B}_{S,Q}) & \xrightarrow{\mu} & H^0(Q, B(S)) \xrightarrow{\omega} H^2(Q, \text{Pic}(S)) \end{array}$$

with exact rows; here the top row is the exact sequence [Hue81b, (1.9)], the middle row the sequence (13.26), the bottom row the exact sequence (2.14), the unlabeled arrow from  $H^0(Q, H^2(N, U))$  to  $H^0(Q, B_{\text{fr}}(T|S))$  is induced by the homomorphism (13.22), and the other unlabeled arrows are either the obvious ones or have been introduced before. If, furthermore, the group  $Q$  is a finite group, the above diagram is available with  $\text{XB}_{\text{fr}}(T|S; G, Q)$  substituted for  $k\mathcal{R}ep(Q, \mathcal{B}_{T|S;G,Q})$  and  $\text{XB}(S, Q)$  for  $k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$ .

*Remark 13.9.* The exact sequences (13.15) and (13.26) are presumably related with an equivariant Amitsur cohomology spectral sequence of the kind given in [Chi72, Sections 1 and 2] and [CR65, Theorem 7.3 p. 61] in the same way as the exact sequences (7.4) and [Hue81b, (1.9)] are related with the spectral sequence associated to a group extension and a module over the extension group, cf. also [Hue81a].

## 14 Examples

Explicit examples of a non-trivial Teichmüller cocycle can be found in, e. g., [CG01], [Hür82], [Mac48b].

Another class of examples arises as follows, cf. Examples 1.1 and 6.1 above: Let  $S$  be a Dedekind domain,  $K$  its quotient field,  $Q$  a group that acts on  $S$  by automorphisms of  $S$ , hence of  $K$ . Then  $U(S)$  is the ordinary group of invertible elements in  $S$  and  $\text{Pic}(S)$  is canonically isomorphic to the ideal class group of  $S$ , the group of fractional ideals modulo that of principal ideals. In particular, in the number field case, the Galois module structure of groups like  $U(S)$  and  $\text{Pic}(S)$  is delicate, cf., e. g., [Frö83], and the calculation of the relevant group cohomology groups is not an easy matter. More calculations are called for in this area.

*Remark 14.1.* In [Hac94], the Teichmüller cocycle serves as a crucial means for building a Galois theory of skew fields. It is worthwhile noting that, in “non-commutative Galois theory”, a counterexample in [Tei40, p. 141] serves as well as a counterexample in [Hac87, p. 558], [Hoc50, p. 298], and [Jac64, §VI.11 p. 147].

## 15 Appendix

As a service to the reader, we recollect some more material from the theory of stably graded symmetric monoidal categories [FW71b], [FW74], [FW00] and use it to illustrate some of the constructions in the present paper.

Recall that an  $S$ -*progenerator* is a faithful finitely generated projective  $S$ -module. Given two  $Q$ -equivariant Azumaya  $S$ -algebras  $(A, \tau_A)$  and  $(B, \tau_B)$ , a  $(B, A, Q)$ -*bimodule*  $(M, \tau_M)$  is a  $(B, A)$ -*bimodule*  $M$  together with an  $S^t Q$ -module structure  $\tau_M: Q \rightarrow \text{Aut}(M)$  which is compatible with the  $Q$ -equivariant structures  $\tau_A: Q \rightarrow \text{Aut}(A)$  and  $\tau_B: Q \rightarrow \text{Aut}(B)$  in the sense that

$${}^x(bya) = {}^x b {}^x y {}^x a, \quad x \in Q, a \in A, b \in B. \quad (15.1)$$

The object  $(S, Q, \kappa_Q)$  of the category *Change* being given, let  $(T, G, \kappa_G)$  be another object of *Change*, and let  $(f, \varphi): (S, Q, \kappa_Q) \rightarrow (T, G, \kappa_G)$  be a morphism in *Change* having  $\varphi: G \rightarrow Q$  surjective, cf. Subsection 2.7 above.

### 15.1 Examples of symmetric monoidal categories

- $\mathcal{Mod}_S$ : the category of  $S$ -modules, a symmetric monoidal category under the operation of tensor product, with  $S$  as unit object, and  $U(\mathcal{Mod}_S) = U(S)$ ;
- $\mathcal{Gen}_S$  [FW71b, §2 p. 17], [FW74, p. 229], [FW00, §2]: the symmetric monoidal subcategory of  $\mathcal{Mod}_S$ , necessarily a groupoid, whose objects are the  $S$ -*progenerators*, with morphisms only the invertible ones, having  $S$  as its unit object and  $U(\mathcal{Gen}_S) = U(S)$  as its unit group;
- $\mathcal{Pic}_S$ : the symmetric monoidal subcategory of  $\mathcal{Gen}_S$ , necessarily group-like, of invertible modules, written in [FW71b, §2 p. 17], [FW00, §2] as  $\mathcal{C}_R$ , reproduced in Subsection 2.6 above;

- $\mathcal{Az}_S$ : the symmetric monoidal subcategory of  $\mathcal{Gen}_S$ , necessarily a groupoid, having the Azumaya  $S$ -algebras as objects, invertible algebra morphisms between Azumaya  $S$ -algebras as morphisms, the ground ring  $S$  as its unit object and unit group  $U(\mathcal{Az}_S)$  trivial [FW71b, §2 p. 18], [FW74, p. 229], [FW00, §2];
- $\mathcal{XAz}_S$ : the quotient category of  $\mathcal{Az}_S$ , necessarily a groupoid, having the same objects as  $\mathcal{Az}_S$ , with morphisms  $A \rightarrow B$  between two objects  $A$  and  $B$  equivalence classes of morphisms  $h: A \rightarrow B$  in  $\mathcal{Az}_S$  under the equivalence relation  $h_1 \sim h_2: A \rightarrow B$  if  $h_1 = h_2 \circ I_a$  for some  $a \in U(A)$  [FW71b, §5 p. 43], [FW00, §2], the notation  $I_a$  referring to the inner automorphism of  $A$  induced by  $a \in U(A)$ ; this category has  $S$  as its unit object, and its unit group  $U(\mathcal{XAz}_S)$  is trivial;
- $\mathcal{B}_S$ , the *Brauer category* of the commutative ring  $S$ , reproduced in Subsection 2.2 above;
- with respect to the ring homomorphism  $f: S \rightarrow T$ , with the obvious interpretations, the relative categories  $\mathcal{Mod}_{T|S}$ ,  $\mathcal{Gen}_{T|S}$ ,  $\mathcal{Pic}_{T|S}$ ,  $\mathcal{Az}_{T|S}$ ,  $\mathcal{XAz}_{T|S}$ , taken as full subcategories of, respectively,  $\mathcal{Mod}_S$ ,  $\mathcal{Gen}_S$ ,  $\mathcal{Pic}_S$ ,  $\mathcal{Az}_S$ ,  $\mathcal{XAz}_S$ ;
- $\mathcal{B}_{T|S}$ , with respect to the ring homomorphism  $f: S \rightarrow T$ , the *relative Brauer category*, introduced in Subsection 13.3.2 above;
- $\mathcal{EB}_{S,Q}$ , the equivariant Brauer category  $\mathcal{EB}_{S,Q}$  of  $S$  relative to the given action of  $Q$  on  $S$ , written as  $\mathcal{B}(R, \Gamma)$  in [FW71b, §5 p. 41] and [FW00, §3]; its objects are the  $Q$ -equivariant Azumaya algebras  $(A, \tau)$ ; a *morphism*  $[(M, \tau_M)]: (A, \tau_A) \rightarrow (B, \tau_B)$  in  $\mathcal{EB}_{S,Q}$  between two given  $Q$ -equivariant Azumaya algebras  $(A, \tau_A)$  and  $(B, \tau_B)$ , necessarily an isomorphism in  $\mathcal{EB}_{S,Q}$ , is an isomorphism class  $[(M, \tau_M)]$  of a  $(B, A, Q)$ -bimodule  $(M, \tau_M: Q \rightarrow \text{Aut}(M))$  whose underlying  $(B, A)$ -bimodule  $M$  is invertible; the operations of tensor product and that of assigning to a  $Q$ -equivariant Azumaya  $S$ -algebra its opposite algebra (as a  $Q$ -equivariant Azumaya  $S$ -algebra) turn  $\mathcal{EB}_{S,Q}$  into a group-like symmetric monoidal category having  $(S, \kappa_Q)$  as its unit object and  $U(\mathcal{EB}_{S,Q}) = \text{EPic}(S)$  as its unit group [FW00, §3], [FW00, Proposition 3.1].
- $\mathcal{EB}_{T|S;G,Q}$ , the *relative equivariant Brauer category associated to the morphism*  $(f, \varphi)$  in *Change*; it has as its objects the  $Q$ -equivariant Azumaya algebras  $(A, \tau)$  such that the  $G$ -equivariant Azumaya algebra  $(T \otimes A, \tau^G)$  that arises by scalar extension has its underlying central  $T$ -algebra  $T \otimes A$  isomorphic to a matrix algebra; given two  $Q$ -equivariant Azumaya algebras  $(A, \tau_A)$  and  $(B, \tau_B)$  in  $\mathcal{EB}_{S,Q}$ , a *morphism*  $(A, \tau_A) \rightarrow (B, \tau_B)$  in the associated precategory  $\mathcal{PreEB}_{T|S;G,Q}$ , necessarily an isomorphism in  $\mathcal{EB}_{T|S;G,Q}$ , is a morphism  $[M, \tau_M]: (A, \tau_A) \rightarrow (B, \tau_B)$  in  $\mathcal{EB}_{S,Q}$ , that is, an isomorphism class of a  $(B, A, Q)$ -bimodule  $(M, \tau_M: Q \rightarrow \text{Aut}(M))$  whose underlying  $(B, A)$ -bimodule  $M$  is invertible, such that, furthermore, the resulting  $T^tG$ -module  $T \otimes M$  is free as a  $T$ -module. We then take  $\mathcal{EB}_{T|S;G,Q}$  to be the resulting subcategory of  $\mathcal{EB}_{S,Q}$  generated by  $\mathcal{PreEB}_{T|S;G,Q}$ , that is, we define morphisms and composition of morphisms as finite strings in  $\mathcal{EB}_{S,Q}$ , of morphisms in  $\mathcal{PreEB}_{T|S;G,Q}$ , and we define the monoidal structure, the operation of inverse, and the unit object as in  $\mathcal{EB}_{S,Q}$ . The resulting category  $\mathcal{EB}_{T|S;G,Q}$  is a group-like symmetric monoidal category and has  $U(\mathcal{EB}_{T|S;G,Q}) = \text{EPic}(T|S)$ .

## 15.2 Examples of stably $Q$ -graded symmetric monoidal categories

- $\mathcal{Mod}_{S,Q}$ , a stably  $Q$ -graded symmetric monoidal category that arises from  $\mathcal{Mod}_S$  as follows: Given two  $S$ -modules  $M$  and  $N$ , a *morphism*  $M \rightarrow N$  of  $S$ -modules of grade  $x \in Q$  is a pair  $(\varphi, x)$  having  $\varphi: M \rightarrow N$  a morphism over  $R = S^Q$  such that  $\varphi(sy) = ({}^x s)y$  ( $s \in S$ ,  $y \in M$ ) [FW74, p. 229], [FW00, §2].

Enhancing each of the categories  $\mathcal{C} = \mathcal{G}en_S, \mathcal{P}ic_S, \mathcal{A}z_S, \mathcal{X}\mathcal{A}z_S, \mathcal{B}_S$  in Subsection 15.1 above to a stably  $Q$ -graded symmetric monoidal category  $\mathcal{C}_Q$  in the same way as enhancing the category  $\mathcal{M}od_S$  of  $S$ -modules to the stably  $Q$ -graded symmetric monoidal category  $\mathcal{M}od_{S,Q}$  just explained yields the following stably  $Q$ -graded symmetric monoidal categories:

- $\mathcal{G}en_{S,Q}$ , written in [FW00] as  $\mathcal{G}en_R$ ;
- $\mathcal{P}ic_{S,Q}$ , written in [FW00, §3] as  $\mathcal{C}_R$ , reproduced in Subsection 2.6 above;
- $\mathcal{A}z_{S,Q}$ , written in [FW00, §2] as  $\mathcal{A}z_R$ ;
- $\mathcal{X}\mathcal{A}z_{S,Q}$ , written in [FW71b, §5 p. 43] as  $Q - \widetilde{\mathcal{A}z_R}$  and in [FW00, §2] as  $Q\mathcal{A}z_R$  (beware: the notation  $Q$  in [op. cit.] has nothing to do with our notation  $Q$  for a group, and the tilde-notation in [FW71b, §5 p. 43] refers to the additional structure of a twisting and need not concern us here); morphisms are now enhanced via the  $Q$ -grading, that is to say, a morphism  $([h], x): A \rightarrow B$  in  $\mathcal{A}z_{S,Q}$  of grade  $x \in Q$  has  $[h]$  an equivalence class of an isomorphism  $h: A \rightarrow B$  of algebras over  $R = S^Q$  such that  $(h, x)$  is, furthermore, a morphism in  $\mathcal{M}od_{S,Q}$  of grade  $x \in Q$ ;
- $\mathcal{B}_{S,Q}$ , the stably  $Q$ -graded Brauer category associated to the commutative ring  $S$  and the  $Q$ -action  $\kappa_Q: Q \rightarrow \text{Aut}(S)$  on  $S$ , reproduced in Subsection 2.5 above.

The morphism  $(f, \varphi): (S, Q, \kappa_Q) \rightarrow (T, G, \kappa_G)$  in *Change* having  $\varphi$  surjective being given, similarly to the construction of the category  $\mathcal{B}_{T|S;G,Q}$  in Subsection 13.3 above, for each of the stably  $Q$ -graded symmetric monoidal categories  $\mathcal{C}_{S,Q} = \mathcal{M}od_{S,Q}, \mathcal{G}en_{S,Q}, \mathcal{P}ic_{S,Q}$ , the stably  $Q$ -graded symmetric monoidal category  $\mathcal{C}_{T|S;G,Q}$  is the subcategory that arises from the ambient category  $\mathcal{C}_{S,Q}$  in essentially the same way as  $\mathcal{B}_{T|S;G,Q}$  arises from the ambient category  $\mathcal{B}_{S,Q}$  save that there is no need to pass through a corresponding precategory: The objects of  $\mathcal{C}_{T|S;G,Q}$  are those objects  $C$  of  $\mathcal{C}_{S,Q}$  such that  $T \otimes C$  is free as a  $T$ -module, and  $\mathcal{C}_{T|S;G,Q} = \mathcal{M}od_{T|S;G,Q}, \mathcal{G}en_{T|S;G,Q}, \mathcal{P}ic_{T|S;G,Q}$  is the respective full subcategory of  $\mathcal{C}_{S,Q}$ . Likewise, for the stably  $Q$ -graded symmetric monoidal category categories  $\mathcal{C}_{S,Q} = \mathcal{A}z_{S,Q}$  and  $\mathcal{C}_{S,Q} = \mathcal{X}\mathcal{A}z_{S,Q}$ , the stably  $Q$ -graded symmetric monoidal category  $\mathcal{C}_{T|S;G,Q}$  arises as the subcategory that has as its objects Azumaya  $S$ -algebras  $A$  such that  $T \otimes A$  is a matrix algebra over  $T$ , and  $\mathcal{A}z_{T|S;G,Q}$  is the corresponding full subcategory of  $\mathcal{A}z_{S,Q}$  and  $\mathcal{X}\mathcal{A}z_{T|S;G,Q}$  that of  $\mathcal{X}\mathcal{A}z_{S,Q}$ . Now, with  $\mathcal{P}ic_{T|S;G,Q}$  substituted for  $\mathcal{C}_{S,Q}$ , the exact sequence (2.10) yields the exact sequence (13.13).

*Remark 15.1.* For an object of  $\mathcal{G}en_{S,Q}$ , that is, for a faithful finitely generated projective  $S$ -module  $M$ , the group  $\text{Aut}(M, Q)$  introduced in Section 9 above is canonically isomorphic to the group  $\text{Aut}_{\mathcal{G}en_{S,Q}}(M)$ .

### 15.3 The standard constructions revisited

The endomorphism functor  $\mathcal{E}nd: \mathcal{G}en_S \rightarrow \mathcal{A}z_S$  induces an exact sequence

$$0 \longrightarrow \text{Pic}(S) \longrightarrow k\mathcal{G}en_S \xrightarrow{\mathcal{E}nd} k\mathcal{A}z_S \longrightarrow \text{B}(S) \longrightarrow 0 \quad (15.2)$$

of abelian monoids [FW71b, §5 p. 38], [FW74, Introduction], [FW00, §3]. This yields  $\text{Pic}(S)$  as the maximal subgroup of the abelian monoid  $k\mathcal{G}en_S$  and recovers the *standard construction* of  $\text{B}(S)$ , cf. Subsection 3.2 above, as the cokernel of the homomorphism  $\mathcal{E}nd$  of abelian monoids, the cokernel of a morphism of monoids being suitably interpreted (in terms of the associated equivalence relation and “cofinality”, cf. [FW74, §12]). The obvious functor  $\Omega: \mathcal{A}z_S \rightarrow \mathcal{B}_S$  induces the isomorphism  $\text{B}(S) \rightarrow k\mathcal{B}_S$  of abelian groups [FW71b, §5 p. 38], [FW00, §3, Theorem 3.2 (i)] quoted in Subsection 3.2.

Likewise, the endomorphism functor  $\mathcal{E}nd: \mathcal{G}en_{S,Q} \rightarrow \mathcal{A}z_{S,Q}$  induces an exact sequence

$$0 \longrightarrow \text{EPic}(S, Q) \longrightarrow k\mathcal{R}ep(Q, \mathcal{G}en_{S,Q}) \xrightarrow{\text{End}} k\mathcal{R}ep(Q, \mathcal{A}z_{S,Q}) \longrightarrow \text{EB}(S, Q) \longrightarrow 0 \quad (15.3)$$

of abelian monoids [FW71b, §5 p. 38], [FW74, Introduction], [FW00, §3]. This yields  $\text{EPic}(S, Q)$  as the maximal subgroup of the abelian monoid  $k\mathcal{R}ep(Q, \mathcal{G}en_{S,Q})$  and recovers the *standard construction* of  $\text{EB}(S, Q)$ , cf. Section 11 above, as the cokernel of the corresponding homomorphism  $\text{End}$  of abelian monoids. The obvious functor  $\Omega: \mathcal{A}z_{S,Q} \rightarrow \mathcal{B}_{S,Q}$  induces an isomorphism  $\text{EB}(S, Q) \rightarrow k\mathcal{EB}_{S,Q}$  of abelian groups [FW71b, §5 p. 38], [FW00, §3, Theorem 3.2 (i)], that is, equivariant Brauer equivalence is equivalent to equivariant Morita equivalence. Moreover, that obvious functor  $\Omega$  factors as

$$\mathcal{A}z_{S,Q} \xrightarrow{\Omega^{\mathcal{A}z}} \mathcal{X}\mathcal{A}z_{S,Q} \xrightarrow{\Omega^{\mathcal{X}\mathcal{A}z}} \mathcal{B}_{S,Q}, \quad (15.4)$$

and the functor  $\Omega^{\mathcal{X}\mathcal{A}z}: \mathcal{X}\mathcal{A}z_{S,Q} \rightarrow \mathcal{B}_{S,Q}$  induces the injection  $\theta: \text{XB}(S, Q) \rightarrow k\mathcal{R}ep(Q, \mathcal{B}_{S,Q})$  of abelian groups spelled out as (10.4) above.

Recall that, given a stably  $Q$ -graded category  $\mathcal{C}_Q$ , the notation  $k_Q\mathcal{C}_Q$  refers to the monoid  $k\mathcal{C} = k\mathcal{K}er(\mathcal{C}_Q) = k\mathcal{C}_Q$ , viewed as a  $Q$ -monoid, cf. Subsection 2.4 above. The functor  $\mathcal{E}nd: \mathcal{G}en_{S,Q} \rightarrow \mathcal{A}z_{S,Q}$  induces, furthermore, a homomorphism

$$\text{H}^0(Q, k_Q\mathcal{G}en_{S,Q}) \longrightarrow k\mathcal{R}ep(Q, \mathcal{X}\mathcal{A}z_{S,Q}) \quad (15.5)$$

of monoids [FW00, §3]. Indeed, let  $M$  be an object of  $\mathcal{G}en_{S,Q}$ . By construction, the grading homomorphism  $\text{Aut}_{\mathcal{G}en_{S,Q}}(M) \rightarrow Q$  is surjective if and only if the isomorphism class of  $M$  in  $k\mathcal{G}en_{S,Q}$  is fixed under  $Q$ . Hence an object  $M$  of  $\mathcal{G}en_{S,Q}$  whose isomorphism class in  $k\mathcal{G}en_{S,Q}$  is fixed under  $Q$  determines the exact sequence

$$1 \longrightarrow \text{Aut}_S(M) \longrightarrow \text{Aut}_{\mathcal{G}en_{S,Q}}(M) \longrightarrow Q \longrightarrow 1, \quad (15.6)$$

plainly congruent to the exact sequence (9.4); in particular, the group  $\text{Aut}_{\mathcal{G}en_{S,Q}}(M)$  is canonically isomorphic to the group  $\text{Aut}(M, Q)$ , cf. (9.3). Now, for any object  $M$  of  $\mathcal{G}en_S$ , the groups  $\text{Aut}_{\mathcal{G}en_S}(M)$ ,  $\text{Aut}_S(M)$ , and  $\text{U}(\text{End}_S(M))$  coincide, and the induced action of  $\text{Aut}_{\mathcal{G}en_{S,Q}}(M)$  on  $\text{End}_S(M)$  yields a commutative diagram of the kind

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Aut}_S(M) & \longrightarrow & \text{Aut}_{\mathcal{G}en_{S,Q}}(M) & \longrightarrow & Q \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ & & \text{U}(\text{End}_S(M)) & \longrightarrow & \text{Aut}(\text{End}_S(M), Q) & \longrightarrow & \text{Out}(\text{End}_S(M), Q) \longrightarrow 1 \end{array} \quad (15.7)$$

and hence an induced  $Q$ -normal structure  $Q \rightarrow \text{Out}(\text{End}_S(M))$  on the split algebra  $\text{End}_S(M)$ . Thus the endomorphism functor  $\mathcal{E}nd: \mathcal{G}en_{S,Q} \rightarrow \mathcal{A}z_{S,Q}$  induces an exact sequence

$$0 \longrightarrow \text{H}^0(Q, \text{Pic}(S)) \longrightarrow \text{H}^0(Q, k_Q\mathcal{G}en_{S,Q}) \xrightarrow{\text{End}} k\mathcal{R}ep(Q, \mathcal{X}\mathcal{A}z_{S,Q}) \longrightarrow \text{XB}(S, Q) \longrightarrow 0 \quad (15.8)$$

of abelian monoids [FW00, §3] which, in turn, recovers the *standard construction* of the crossed Brauer group  $\text{XB}(S, Q)$  of  $S$  relative to  $Q$  given Section 10.1 above. The unit object of  $\mathcal{X}\mathcal{A}z_{S,Q}$  is represented by  $(S, \kappa_Q)$ . This kind of construction is given in [FW71b, Theorem 4 p. 43], [FW00, Section 3, a few lines before Theorem 3.2] (the cokernel of  $\text{End}$  being written

as  $QB(R, \Gamma)$ ). In general, for the “crossed” versions, the equivalence between Brauer and Morita equivalence persists only when the group  $Q$  is finite, that is the canonical homomorphism  $\theta: \text{XB}(S, Q) \rightarrow k\mathcal{R}ep(Q, \mathcal{B}_{S, Q})$  of abelian groups given as (10.4) above is injective, see Theorem 10.10 (i) above, but to prove that  $\theta$  is surjective we need the additional hypothesis that  $Q$  be a finite group, see Theorem 10.10 (iii) above.

The above constructions, applied, with respect to the morphism  $(f, \varphi)$  in *Change*, to the functors  $\mathcal{E}nd: \mathcal{G}en_{T|S} \rightarrow \mathcal{A}z_{T|S}$  and  $\mathcal{E}nd: \mathcal{G}en_{T|S; G, Q} \rightarrow \mathcal{A}z_{T|S; G, Q}$ , yield the exact sequences

$$\begin{aligned} 0 &\longrightarrow \text{Pic}(T|S) \longrightarrow k\mathcal{G}en_{T|S} \xrightarrow{\mathcal{E}nd} k\mathcal{A}z_{T|S} \longrightarrow \text{B}_{\text{fr}}(T|S) \longrightarrow 0 \\ 0 &\longrightarrow \text{EPic}(T|S, Q) \longrightarrow k\mathcal{R}ep(Q, \mathcal{G}en_{T|S; G, Q}) \xrightarrow{\mathcal{E}nd} k\mathcal{R}ep(Q, \mathcal{A}z_{T|S; G, Q}) \longrightarrow \text{EB}_{\text{fr}}(T|S; G, Q) \longrightarrow 0 \\ 0 &\longrightarrow \text{H}^0(Q, \text{Pic}(T|S)) \longrightarrow \text{H}^0(Q, k_Q \mathcal{G}en_{T|S; G, Q}) \xrightarrow{\mathcal{E}nd} k\mathcal{R}ep(Q, \mathcal{X}\mathcal{A}z_{T|S; G, Q}) \longrightarrow \text{XB}_{\text{fr}}(T|S; G, Q) \longrightarrow 0 \end{aligned}$$

of abelian monoids. These recover the standard constructions of the abelian groups  $\text{B}_{\text{fr}}(T|S)$ ,  $\text{EB}_{\text{fr}}(T|S; G, Q)$ , and  $\text{XB}_{\text{fr}}(T|S; G, Q)$ , cf. Subsection 13.3.1 above.

## References

- [AG60a] Maurice Auslander and Oscar Goldman. The Brauer group of a commutative ring. *Trans. Amer. Math. Soc.*, 97:367–409, 1960.
- [AG60b] Maurice Auslander and Oscar Goldman. Maximal orders. *Trans. Amer. Math. Soc.*, 97:1–24, 1960.
- [Aus66] Bernice Auslander. The Brauer group of a ringed space. *J. Algebra*, 4:220–273, 1966.
- [Bas68] Hyman Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.
- [BHS11] Ronald Brown, Philip J. Higgins, and Rafael Sivera. *Nonabelian algebraic topology*, volume 15 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011. Filtered spaces, crossed complexes, cubical homotopy groupoids, With contributions by Christopher D. Wensley and Sergei V. Soloviev.
- [Bou61] N. Bourbaki. *Éléments de mathématique. Fascicule XXVII. Algèbre commutative. Chapitre 1: Modules plats. Chapitre 2: Localisation*. Actualités Scientifiques et Industrielles, No. 1290. Herman, Paris, 1961.
- [Bru66] Armand Brumer. Galois groups of extensions of algebraic number fields with given ramification. *Michigan Math. J.*, 13:33–40, 1966.
- [CG01] Antonio M. Cegarra and Antonio R. Garzón. Obstructions to Clifford system extensions of algebras. *Proc. Indian Acad. Sci. Math. Sci.*, 111(2):151–161, 2001.
- [CG06] A. M. Cegarra and A. R. Garzón. Equivariant Brauer groups and cohomology. *J. Algebra*, 296(1):56–74, 2006.

- [CGG00] A. M. Cegarra, A. R. Garzón, and A. R. Grandjean. Graded extensions of categories. *J. Pure Appl. Algebra*, 154(1-3):117–141, 2000. Category theory and its applications (Montreal, QC, 1997).
- [CGO01] A. M. Cegarra, A. R. Garzón, and J. A. Ortega. Graded extensions of monoidal categories. *J. Algebra*, 241(2):620–657, 2001.
- [Chi72] L. N. Childs. On normal Azumaya algebras and the Teichmüller cocycle map. *J. Algebra*, 23:1–17, 1972.
- [CHR65] S. U. Chase, D. K. Harrison, and Alex Rosenberg. Galois theory and Galois cohomology of commutative rings. *Mem. Amer. Math. Soc. No.*, 52:15–33, 1965.
- [CR65] S. U. Chase and Alex Rosenberg. Amitsur cohomology and the Brauer group. *Mem. Amer. Math. Soc. No.*, 52:34–79, 1965.
- [Deu36] Max Deuring. Einbettung von Algebren in Algebren mit kleinerem Zentrum. *J. Reine Angew. Math.*, 175:124–128, 1936.
- [EM48] Samuel Eilenberg and Saunders MacLane. Cohomology and Galois theory. I. Normality of algebras and Teichmüller’s cocycle. *Trans. Amer. Math. Soc.*, 64:1–20, 1948.
- [Frö83] Albrecht Fröhlich. *Galois module structure of algebraic integers*, volume 1 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1983.
- [FW71a] A. Fröhlich and C. T. C. Wall. Equivariant Brauer groups in algebraic number theory. In *Colloque de Théorie des Nombres (Univ. de Bordeaux, Bordeaux, 1969)*, pages 91–96. Bull. Soc. Math. France, Mém No. 25. Soc. Math. France, Paris, 1971.
- [FW71b] A. Fröhlich and C.T.C. Wall. Generalisations of the Brauer group. I. 1971. Preprint.
- [FW74] A. Fröhlich and C. T. C. Wall. Graded monoidal categories. *Compositio Math.*, 28:229–285, 1974.
- [FW00] A. Fröhlich and C. T. C. Wall. Equivariant Brauer groups. In *Quadratic forms and their applications (Dublin, 1999)*, volume 272 of *Contemp. Math.*, pages 57–71. Amer. Math. Soc., Providence, RI, 2000.
- [Hac87] Michel Hacque. Théorie de Galois des anneaux presque-simples. *J. Algebra*, 108(2):534–577, 1987.
- [Hac94] Michel Hacque. Structure globale des extensions régulières galoisiennes. *Comm. Algebra*, 22(2):611–674, 1994.
- [Hat79] Akira Hattori. On groups  $H^n(S/R)$  related to the Amitsur cohomology and the Brauer group of commutative rings. *Osaka J. Math.*, 16(2):357–382, 1979.
- [Hoc50] G. Hochschild. Automorphisms of simple algebras. *Trans. Amer. Math. Soc.*, 69:292–301, 1950.

- [Hol79] D. F. Holt. An interpretation of the cohomology groups  $H^n(G, M)$ . *J. Algebra*, 60(2):307–318, 1979.
- [HS53] G. Hochschild and J.-P. Serre. Cohomology of group extensions. *Trans. Amer. Math. Soc.*, 74:110–134, 1953.
- [Hue80] Johannes Huebschmann. Crossed  $n$ -fold extensions of groups and cohomology. *Comment. Math. Helv.*, 55(2):302–313, 1980.
- [Hue81a] Johannes Huebschmann. Automorphisms of group extensions and differentials in the Lyndon-Hochschild-Serre spectral sequence. *J. Algebra*, 72(2):296–334, 1981.
- [Hue81b] Johannes Huebschmann. Group extensions, crossed pairs and an eight term exact sequence. *J. Reine Angew. Math.*, 321:150–172, 1981.
- [Hür82] Werner Hürlimann. Brauer group and Diophantine geometry: a cohomological approach. In *Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981)*, volume 917 of *Lecture Notes in Math.*, pages 43–65. Springer, Berlin-New York, 1982.
- [Jac64] Nathan Jacobson. *Structure of rings*. American Mathematical Society Colloquium Publications, Vol. 37. Revised edition. American Mathematical Society, Providence, R.I., 1964.
- [Kan64] Teruo Kanzaki. On commutator rings and Galois theory of separable algebras. *Osaka J. Math.* 1 (1964), 103-115; correction, *ibid.*, 1:253, 1964.
- [Kan68] Teruo Kanzaki. On generalized crossed product and Brauer group. *Osaka J. Math.*, 5:175–188, 1968.
- [Knu75] M.-A. Knus. A Teichmüller cocycle for finite extensions. 1975. ETH-preprint.
- [Mac48a] Saunders MacLane. A nonassociative method for associative algebras. *Bull. Amer. Math. Soc.*, 54:897–902, 1948.
- [Mac48b] Saunders MacLane. Symmetry of algebras over a number field. *Bull. Amer. Math. Soc.*, 54:328–333, 1948.
- [Mac67] Saunders MacLane. *Homology*. Springer-Verlag, Berlin, first edition, 1967. Die Grundlehren der mathematischen Wissenschaften, Band 114.
- [Mac71] Saunders MacLane. *Categories for the working mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.
- [Mac79] Saunders MacLane. Historical note. *J. Algebra*, 60(2):319–320, 1979. Appendix to [Hol79].
- [Nak53] Tadasi Nakayama. On a 3-cohomology class in class field theory and the relationship of algebra- and idèle-classes. *Ann. of Math. (2)*, 57:1–14, 1953.
- [Par64] Bodo Pareigis. Über normale, zentrale, separable Algebren und Amitsur-Kohomologie. *Math. Ann.*, 154:330–340, 1964.



- [RZ61] Alex Rosenberg and Daniel Zelinsky. Automorphisms of separable algebras. *Pacific J. Math.*, 11:1109–1117, 1961.
- [Tat67] J. T. Tate. Global class field theory. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 162–203. Thompson, Washington, D.C., 1967.
- [Tay53] Robert L. Taylor. Compound group extensions. I. Continuations of normal homomorphisms. *Trans. Amer. Math. Soc.*, 75:106–135, 1953.
- [Tei40] Oswald Teichmüller. Über die sogenannte nichtkommutative Galoissche Theorie und die Relation  $\xi_{\lambda,\mu,\nu}\xi_{\lambda,\mu\nu,\pi}\xi_{\mu,\nu,\pi}^\lambda = \xi_{\lambda,\mu,\nu\pi}\xi_{\lambda,\mu,\nu,\pi}$ . *Deutsche Math.*, 5:138–149, 1940.
- [Ulbr79] K.-H. Ulbrich. An abstract version of the Hattori-Villamayor-Zelinsky sequences. *Sci. Papers College Gen. Ed. Univ. Tokyo*, 29(2):125–137, 1979.
- [Ulbr89] K.-H. Ulbrich. On the Teichmüller cocycle map. *J. Algebra*, 124(2):461–471, 1989.
- [Ulbr94] K.-H. Ulbrich. On the Teichmüller cocycle map and a theorem of Eilenberg-Mac Lane. *Bull. Sci. Math.*, 118(3):307–324, 1994.
- [VZ78] O. E. Villamayor and D. Zelinsky. Brauer groups and Amitsur cohomology for general commutative ring extensions. *J. Pure Appl. Algebra*, 10(1):19–55, 1977/78.
- [Whi49] J. H. C. Whitehead. Combinatorial homotopy. II. *Bull. Amer. Math. Soc.*, 55:453–496, 1949.
- [Yok78] Kenji Yokogawa. An application of the theory of descent to the  $S \otimes_R S$ -module structure of  $S/R$ -Azumaya algebras. *Osaka J. Math.*, 15(1):21–31, 1978.
- [Zel76] D. Zelinsky. Long exact sequences and the Brauer group. In *Brauer groups (Proc. Conf., Northwestern Univ., Evanston, Ill., 1975)*, pages 63–70. Lecture Notes in Math., Vol. 549. Springer, Berlin, 1976.